(14 pts) **1.** For each of the following, circle **T** if the statement is always true and circle **F** if it can be false.
Do not guess: wrong answers will receive **−2 marks**.

(a) If A is a 3×4 matrix, then the system $A\mathbf{x} = \mathbf{0}$ always has infinitely many solutions. **T** **F**
Solution: True: after row-reducing, at most three leading ones, but four columns.

(b) A set of vectors in $\mathbb{R}^n$ that contains $\mathbf{0}$ is linearly independent. **T** **F**
Solution: False: see Example 9, p. 128

(c) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{0}$, then so is $\mathbf{x}_1 + \mathbf{x}_2$. **T** **F**
Solution: True: $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0}$.

(d) If A is a matrix with two identical rows, then $\det(A) = 0$. **T** **F**
Solution: True: Theorem 4.2.3(a), p. 186

(e) If A is invertible, then so is $\text{adj}(A)$. **T** **F**
Solution: True: $A \cdot \text{adj}(A) = \det(A)I$, so $(1/\det(A))A = (\text{adj}(A))^{-1}$.

(f) If $\lambda = 1$ is an eigenvalue of A, then A is invertible. **T** **F**
Solution: False: e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not invertible.

(g) The set of solutions to the equation $5x + 6y + 7z = 8$ is a subspace of $\mathbb{R}^3$. **T** **F**
Solution: False: in particular, $(0, 0, 0)$ is not a solution.

For all remaining problems, you must show all of your work and explain fully to receive full credit.

2. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}$.

(2 pts) (a) Find $\|\mathbf{v}\|$.
Solution:

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{0^2 + 3^2 + (-4)^2} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$

(2 pts) (b) Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.
Solution:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (-1, 1, 0) \cdot (0, 3, -4) \\ &= (-1)0 + (1)3 + 0(-4) \\ &= 3. \end{aligned}$$

(3 pts) (c) Compute the cross product $\mathbf{u} \times \mathbf{v}$.
Solution:

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left(\begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix}, -\begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix}, \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} \right) \\ &= (-4, -4, -3). \end{aligned}$$

(4 pts) **3.** (a) Use row operations to put the following matrix into reduced row echelon form. You must show the intermediate matrices, but you don't have to describe the row operations you used.

$$A = \begin{bmatrix} 2 & 4 & 6 & 2 \\ -1 & -1 & -2 & 1 \\ 2 & 5 & 7 & 4 \end{bmatrix}$$

Solution: Exchange rows 1 and 2, then multiply the first row by -1 to get

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 4 & 6 & 2 \\ 2 & 5 & 7 & 4 \end{bmatrix}$$

Then subtract 2 times row 1 from rows 2 and 3:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

Now divide row 2 by 2, and subtract three times row 2 from row 3:

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Last, subtract row 2 from row 1:

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3 pts) (b) Using the previous part, write down the general solution to the following system in parametric form.

$$\begin{aligned} 2x + 4y + 6z &= 2 \\ -x - \ y - 2z &= 1 \\ 2x + 5y + 7z &= 4 \end{aligned}$$

Solution: Having row-reduced the augmented matrix to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

we see z is a free variable. Let $z = t$; then $x = -3 - t$, $y = -2 - t$, and $z = t$ is the parametric solution.

4. Let A and B be 3×3 matrices such that $\mathrm{tr}(A) = 3$ and $\mathrm{tr}(B) = 5$.

(2 pts) (a) Compute $\mathrm{tr}(A + 2B)$.
Solution:

$$\begin{aligned} \mathrm{tr}(A + 2B) &= \mathrm{tr}(A) + \mathrm{tr}(2B) \\ &= \mathrm{tr}(A) + 2\mathrm{tr}(B) \\ &= 3 + 2(5) \\ &= 13 \end{aligned}$$

(2 pts) (b) Compute $\mathrm{tr}(A^T)$.
Solution: A matrix has the same trace as its transpose, so $\mathrm{tr}(A^T) = 3$.

(4 pts) **5.** Determine whether or not the vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ are linearly independent.

Solution: One way is to check whether or not the determinant of $[\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3]$ is nonzero. Expanding by cofactors along the first row,

$$\begin{aligned} \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 7 \end{bmatrix} &= 2\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 2(7-6) - (7-4) + (3-2) \\ &= 0 \end{aligned}$$

so the vectors are linearly dependent.

6. Let A and B be 3×3 matrices such that $\det(A) = 2$ and $\det(B) = 4$. Compute:

(2 pts) (a) $\det(2A)$

Solution: $\det(2A) = 2^3 \det(A) = 16$.

(2 pts) (b) $\det(B^T)$

Solution: The determinant of a matrix is the same as that of its transpose, so $\det(B^T) = 4$ too.

(2 pts) (c) $\det(AB)$

Solution: $\det(AB) = \det(A) \cdot \det(B) = 2 \cdot 4 = 8$.

(3 pts) **7.** Find an elementary matrix E for which $B = EA$, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -1 & -1 \\ -2 & -5 & 6 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -1 & -1 \\ 0 & -1 & 4 & 3 \end{bmatrix}.$$

Solution: By inspection, B is obtained from A by adding two times row 1 to row 3. So E is obtained by doing the same row operation to the identity matrix, which means

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

(4 pts) **8.** Use Cramer's rule to find y without solving for x and z:

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 6 \\ 7x + 5y + z &= 7 \end{aligned}$$

Solution: By Cramer's rule,

$$\begin{aligned} y &= \frac{\det \begin{bmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 7 & 7 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 7 & 5 & 1 \end{bmatrix}} \\ &= \frac{6}{1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} \\ &= \frac{6}{-2} \\ &= -3. \end{aligned}$$

(4 pts) **9.** For what values of k, if any, is the vector $\mathbf{b}$ in the column space of the matrix A, where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \\ -1 & 0 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ k-2 \\ 1 \end{bmatrix}?$$

Solution: We know that $\mathbf{b}$ is in the column space of A if and only if $A\mathbf{x} = \mathbf{b}$ has a solution. So row-reduce the augmented matrix $(A|\mathbf{b})$ to see when this is the case:

$$\begin{aligned} (A|\mathbf{b}) &= \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & k-2 \\ -1 & 0 & -3 & 1 \end{array}\right] \\ &\to \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & k-2 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\to \left[\begin{array}{rrr|r} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

For the equation $A\mathbf{x} = \mathbf{b}$ to have a solution, the k in the third row must be zero. So $\mathbf{b}$ is in the column space of A if and only if $k = 0$.

(4 pts) **10.** (a) Find the eigenvalues of $A = \begin{bmatrix} -2 & 4 \\ 3 & 2 \end{bmatrix}$ and their algebraic multiplicities.

Solution: Compute the characteristic polynomial,

$$\begin{aligned} \det(\lambda \cdot I_2 - A) &= \det \begin{bmatrix} \lambda+2 & -4 \\ -3 & \lambda-2 \end{bmatrix} \\ &= (\lambda+2)(\lambda-2) - 12 \\ &= \lambda^2 - 16 \\ &= (\lambda-4)(\lambda+4) \end{aligned}$$

The eigenvalues of A the roots of the characteristic polynomial, which we see from above are $\lambda = 4$ and $\lambda = -4$.

(4 pts) (b) Find the eigenspace associated to the smallest eigenvalue you found.

Solution: The smallest eigenvalue is $\lambda = -4$. The eigenspace is the solution space to $(-4I - A)\mathbf{x} = \mathbf{0}$, so row-reduce:

$$\begin{aligned} -4I - A &= \begin{bmatrix} -2 & -4 \\ -3 & -6 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

whose solutions have the form $t(2,1)$. So the eigenspace consists of all multiples of the vector $(2,1)$.

(2 pts) **11.** (a) Find the eigenvalues of the matrix A below, and their algebraic multiplicities.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution: Since the matrix is upper-triangular, the eigenvalues are the diagonal entries. So 2 is an eigenvalue with multiplicity 1, and -1 is an eigenvalue with multiplicity 2.

(3 pts) (b) Find the characteristic polynomial of A^4.

Solution: The eigenvalues of A^4 are $(-1)^4 = 1$ and $2^4 = 16$, with multiplicities 2 and 1, respectively. So the characteristic polynomial of A^4 is $(\lambda - 1)^2(\lambda - 16)$.

(4 pts) **12.** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$.

Solution: Row-reduce the matrix $[A|I_3]$:

$$\begin{aligned}
[A|I_3] &= \left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & -2 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1/2 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 3/2 \\ 0 & 0 & 1 & 0 & 1 & -1/2 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & 1/2 \\ 0 & 1 & 0 & 0 & -2 & 3/2 \\ 0 & 0 & 1 & 0 & 1 & -1/2 \end{array}\right]
\end{aligned}$$

So $A^{-1} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ 0 & -2 & 3/2 \\ 0 & 1 & -1/2 \end{bmatrix}$.