Theory of multidimensional persistence

a brief introduction

2018

Outline

1. Motivation
   - Different filtrations
   - The tasks of mutildimensional persistence

2. Language for multidimensional persistence
   - Persistence module and correspondence
   - Classification

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Different filtrations

We start with a point cloud $X \subseteq \mathbb{R}^n$ which may come from a noisy sampling.

The usual filtration by radius ϵ

Take X as the set of vertices and $VR_\epsilon(X)$ gives a filtration of simplicial complexes.

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Filtration of X by density

Fixing K a positive integer, define $\rho_K(x) = d(x, x_K)$ where x_K is the K-th closest point to x. There is a filtration on X by $r_1 < r_2 < \cdots < r_n$

$$X_{r_1} \subseteq X_{r_2} \subseteq \cdots \subseteq X_{r_n}$$

where

$$X_r = \{x \in X : \rho_K(x) \leq r\}.$$

Reason of applying different filtrations

What's the reason of doing the second filtration?

Figure: Data set used in HDBSCAN

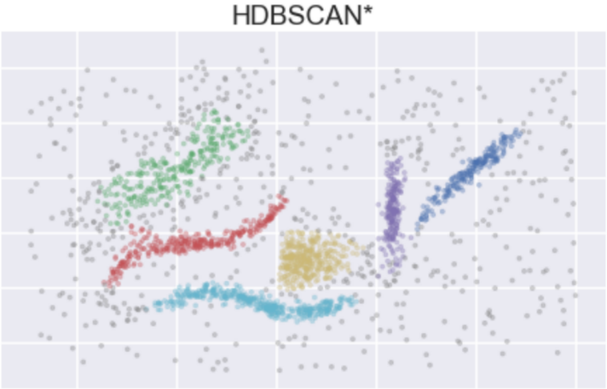

image source: [McInnes and Healy 2017]

More filtrations

Filtration by curvature

How to distinguish a bottle from a glass?

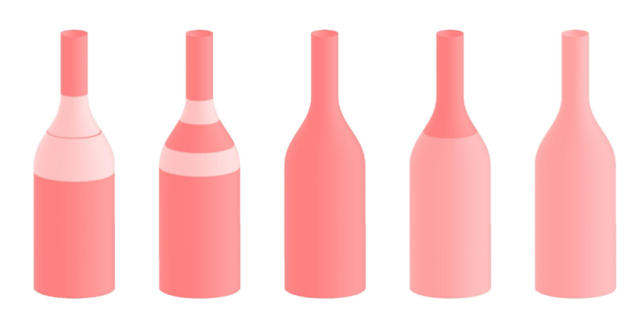

image source: [Carlsson, A. Zomorodian, et al. 2005]

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Three major tasks of persistence homology

We illustrate theses tasks by example in dimension 1.

Correspondence

Persistence homology $\Leftrightarrow$ finitely generated graded $k[t]$-module.

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Classification

Structure of finitely generated graded $k[t]$-module:

$$\left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i}\mathbb{F}[t]\right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j}\mathbb{F}[t]/t^{d_j}\mathbb{F}[t]\right).$$

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Parametrization

The classification gives set of half-infinite intevals $[\alpha_i, \infty)_{i=1}^{n}$ and finite intervals $[\gamma_j, \gamma_j + d_j)_{j=1}^{m}$ called *barcode*.

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Some notations

Definition ($\leq$)

Let $\mathbb{N}$ be the set of natural numbers. For two $u, v \in \mathbb{N}^n$, we say $u \leq v$ if

$$u_i \leq v_i, \; i = 1, \ldots, n$$

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Example

(1) $(0, 1) \leq (1, 1)$.

(2) $(3, 1) \leq (3, 2)$.

(3) $(2, 3) \not\leq (3, 2)$.

Some notations

Definition (*n*-graded rings and modules)

Fixing an $n \in \mathbb{N}$, an *n-graded ring* is a ring R with a decomposition of Abelian groups

$$R \simeq \bigoplus_{v \in \mathbb{N}^n} R_v, \tag{1}$$

$$R_u \cdot R_v \subseteq R_{u+v}, \forall u, v \in \mathbb{N}^n. \tag{2}$$

An *n-graded module* over R is a module M over R with decomposition

$$M \simeq \bigoplus_{v \in \mathbb{N}^n} M_v \tag{3}$$

$$R_u \cdot M_v \subseteq M_{u+v}, \forall u, v \in \mathbb{N}^n. \tag{4}$$

A *homomorphism* of n-grade module over R is an R-module homomorphism that preserves $\mathbb{N}^n$-grading.

n-graded vector space

Definition (n-graded vector spaces)

An *n-graded vector space* V over field k is a k-vector space with the following decomposition of k-vector spaces.

$$V = \bigoplus_{u \in \mathbb{N}^n} V_u.$$

A homomorphism of n-graded vector space is a homomorphism of vector spaces that preserves $\mathbb{N}^n$-grading.

Some notations

Example (n-graded ring and module:)

(1) $R = k[x_1, \ldots, x_n]$ is an n-graded ring.

(2) $M = (x_1, \ldots, x_n)$ is an n-graded module over R.

(3) $V = k$ is an n-graded module over R where

$$V_0 = k,$$
$$V_u = 0, \ \forall u \neq 0$$

(4) Every n-graded module over R is also an n-graded vector space.

(5) For any n-graded module M, we can construct an n-graded vector space

$$k \otimes_R M \simeq M/(x_1, \ldots, x_n)M.$$

Multifiltered space

Definition (Multifiltered space)

A space X is *multifiltered* if there is a family of subspaces $\{X_v \subseteq X\}_{v \in \mathbb{N}^n}$ with

$$X_u \subseteq X_v, \ \forall u \leq v.$$

such that the following diagram commutes for $u \leq v_1, v_2 \leq w$

$$\begin{array}{ccc} X_{v_2} & \longrightarrow & X_w \\ \uparrow & & \uparrow \\ X_u & \longrightarrow & X_{v_1}. \end{array}$$

Multifiltered space

Example (Bifiltered triangle)

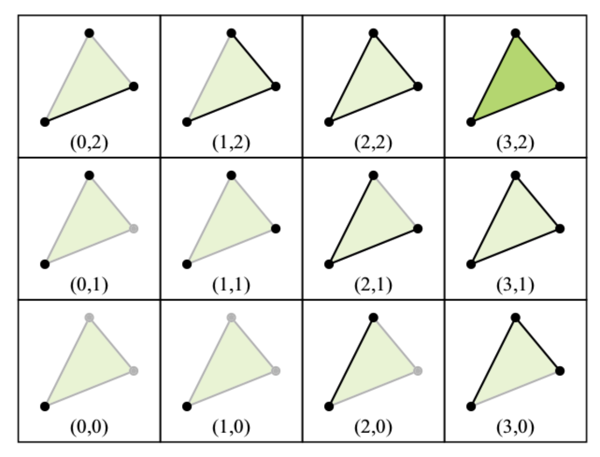

image source: [Carlsson and A. Zomorodian 2009]

Homology of a multifiltered space

Example (cont.)

Taking 0th homology of the previous example, we get

$$
\begin{array}{ccccccc}
k^2 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
k^2 & \longrightarrow & k^3 & \longrightarrow & k & \longrightarrow & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k.
\end{array}
$$

Persistence module

Definition (Persistence module)

A *persistence module* M is a family of k-vector spaces $\{M_v\}_{v \in \mathbb{N}^n}$ together with homomorphisms $\phi_{u,v} : M_u \to M_v$ for all $u \leq v$ such that
$\phi_{u,v} \circ \phi_{v,w} = \phi_{u,w}$ whenever $u \leq v \leq w$.
A persistence module M is called *finite* if every M_v is finite dimensional and $\phi_{u,v} : M_u \to M_v$ are isomorphisms for "large" enough u, v, that is, every coordinate u_i, v_i large enough.

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Structure of a persistence module

Given a persistence module M, we define an n-graded module over $R = k[x_1, \ldots, x_n]$ by

$$\alpha(M) = \bigoplus_{v \in \mathbb{N}^n} M_v$$

and $x^{v-u} : M_u \to M_v$ is just $\phi_{u,v}$ for $u \leq v$.

Persistence module

Correspondence [Carlsson and A. Zomorodian 2009]

The correspondence α defines an equivalence of categories between the category of finite persistence modules over k and the category of finitely generated n-graded modules over $R = k[x_1, \ldots, x_n]$.

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Free n-graded R-module

Definition (multiset)

Let $S \subseteq \mathbb{N}^n$ be a subset and $\mu : S \to \mathbb{N}$ be a function. A *multiset* denoted (S, μ) is a subset of $S \times \mathbb{N}$:

$$(S, \mu) := \{(s, \mu(s)) : s \in S\}.$$

A multiset is called *finite* if μ is non-zero only on finitely many $s \in S$.

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Definition (free n-graded module)

Let ξ be a multiset. The *free n-graded R-module* associated to ξ is the direct sum

$$F(\xi) = \bigoplus_{(u,i)\in\xi} R(u)$$

$\xi = \{(1,0):2, (2,1):1\}$

$F(\xi) = R(1,0)^2 \oplus R(2,1)$

where $R(u)$ is an upward shifting of R by u, that is, $R(u)_v = R_{v-u}$.
A free n-graded R-module determine a unique multiset.

The first two invariants $\xi_0(M)$ and $\xi_1(M)$

To classfiy an arbitrary finitely generated n-graded R-module M, one idea is to introduce some invariants of M. We first look at two invariants.

Step 1

First we form the n-graded vector space

$$k \otimes_R M = M/(x_1, \ldots, x_n)M$$

and get a multiset $\xi_0(M)$ from this vector space.

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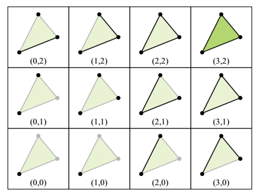
Example



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$\xi_0(M) = \{(0,0):1$
$(0,1):1$
$(1,1):1\}$

$k \otimes_R M =$

0	0	0	0
k	k	0	0
k	0	0	0

The first two invariants $\xi_0(M)$ and $\xi_1(M)$

Step 2

Form the free n-graded R-module $F(\xi_0(M))$. There is epimorphism $\phi : F(\xi_0(M)) \to M$ such that

$$k \otimes_R \phi : k \otimes_R F(\xi_0(M)) \to k \otimes_R M$$

is isomorphism of n-graded vector spaces. Any two such homomorphism differ by an automorphism of $F(\xi_0(M))$.

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Remark

Another view of the above two steps is from minimal free resolution:

(1) Find a set of minimal $\mathbb{N}^n$-homogeneous generators of M.

(2) Let F be a free n-graded module on this set. $\phi : F \twoheadrightarrow M$.

(3) Any two such free module F_1 and F_2 are isomorphic and two epimorphisms differ by an automorphism of F.

The first two invariants $\xi_0(M)$ and $\xi_1(M)$

Step 3

We have

$$0 \to \ker \phi \to F(\xi_0(M)) \xrightarrow{\phi} M \to 0.$$

Let $\xi_1(M)$ be the multiset determined by $k \otimes_R \ker \phi$. $\xi_1(M)$ is independent of the epimorphism $F(\xi_0(M)) \to M$.

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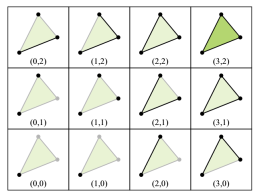
Example



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$k \otimes_R M =$

0	0	0	0
k	k	0	0
k	0	0	0

$\xi_0(M) = \{(0,0) : 1, (0,1) : 1, (1,1) : 1\}$.

$F(\xi_0(M)) = R(0,0) \oplus R(0,1) \oplus R(1,1)$

The first two invariants $\xi_0(M)$ and $\xi_1(M)$

Example (cont.)

$$M = \begin{array}{ccccccc} k^2 & \rightarrow & k & \rightarrow & k & \rightarrow & k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k^2 & \rightarrow & k^3 & \rightarrow & k & \rightarrow & k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \rightarrow & k & \rightarrow & k. \end{array}, \quad F(\xi_0(M)) = \begin{array}{ccccccc} k^2 & \rightarrow & k^3 & \rightarrow & k^3 & \rightarrow & k^3 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k^2 & \rightarrow & k^3 & \rightarrow & k^3 & \rightarrow & k^3 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k. \end{array}$$

$$\ker\phi = \begin{array}{ccccccc} 0 & \rightarrow & k^2 & \rightarrow & k^2 & \rightarrow & k^2 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & 0 & \rightarrow & k^2 & \rightarrow & k^2 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}, \quad \xi_1(M) = \{(1,2) : 2, (2,1) : 2\}$$

The first two invariants $\xi_0(M)$ and $\xi_1(M)$

Example (cont.)

$$M = \begin{array}{ccccccc} k^2 & \to & k & \to & k & \to & k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k^2 & \to & k^3 & \to & k & \to & k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \to & k & \to & k. \end{array} \quad , F(\xi_0(M)) = \begin{array}{ccccccc} k^2 & \to & k^3 & \to & k^3 & \to & k^3 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k^2 & \to & k^3 & \to & k^3 & \to & k^3 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & k & \longrightarrow & k & \longrightarrow & k. \end{array}$$

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Remark

(1) $\xi_0(M)$: The stage and multiplicity of birth of homology classes.

(1) $\xi_1(M)$: The stage and multiplicity of death of homology classes.

Classification

Problem in classification of multidimensional persistence

If $n = 1$, then $\xi_0(M)$ and $\xi_1(M)$ completely classify all graded $k[t]$-modules.
If $n > 1$, $\xi_0(M)$ and $\xi_1(M)$ do not give complete classification.

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If $n > 1$, $\xi_0(M)$ and $\xi_1(M)$ do not give complete classification.

Question

Given multiset ξ_0, ξ_1, what is the set of isomorphism classes of n-graded R-module $[M]$ such that

$$\xi_0(M) = \xi_0,\ \xi_1(M) = \xi_1.$$

Classification

Theorem (Carlsson and A. Zomorodian 2009)

Isomorphism classes of n-graded R-module $[M]$ such that

$$\xi_0(M) = \xi_0,\ \xi_1(M) = \xi_1$$

are one to one corresponding to

$$\left\{ \begin{array}{l} \text{Submodules } K \text{ of } F(\xi_0) \text{ such that} \\ \xi_0(K) = \xi_1. \end{array} \right\} \Big/ GL(F(\xi_0))$$

where $GL(F(\xi_0))$ is the automorphism group of $F(\xi_0)$ and

$$GL(F(\xi_0)) \simeq \prod_{u \in \xi_0} GL(F(\xi_0)_u).$$

It is the orbit of an algebraic variety under an algebraic group action.

A simple example

Example

Let $n = 2$ and

$$\xi_0 = \{(0,0) : 2\}$$
$$\xi_1 = \{(1,0) : 1, (0,1) : 1\}.$$

Submodules K of $F(\xi_0)$ such that $\xi_0(K) = \xi_1$ correspondes to pairs of lines in k^2

$$Gr_1(k^2) \times Gr_1(k^2) = \mathbb{P}^1(k) \times \mathbb{P}^1(k)$$

and the group

$$GL(F(\xi_0)) \simeq GL(k^2)$$

acts on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ componentwise.

A simple example

Example (cont.)

Assume that $(l_1, l_2) \in Gr_1(k^2) \times Gr_1(k^2)$.

(1) If $l_1 = l_2$, then the orbit of (l_1, l_2) consists of one point.

(2) If $l_1 \neq l_2$, then the orbit of (l_1, l_2) consists of one point.

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(2) If $l_1 \neq l_2$, then the orbit of (l_1, l_2) consists of one point.

Example (cont.)

Realization of these two cases as bifiltered spaces: Start with two loops at $(0, 0)$.

(1) Sew cylinders at both stage $(1, 0)$ and $(0, 1)$.

(2) Sew a cylinder at $(1, 0)$ and a Mobius band at $(0, 1)$.

Parametrization: Non-existence of complete discrete invariants

Example (Carlsson and A. Zomorodian 2009)

Let $n = 2$ and

$$\xi_0 = \{(0,0) : 2\}$$
$$\xi_1 = \{(3,0) : 1, (2,1) : 1, (1,2) : 1, (0,3) : 1\}.$$

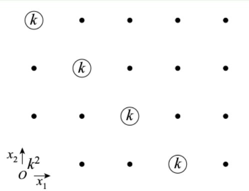

image source: [Carlsson and A. Zomorodian 2009]

Parametrization: Non-existence of complete discrete invariants

Example (cont.)

Submodules K of $F(\xi_0)$ such that $\xi_0(K) = \xi_1$ correspondes to 4-tuple of lines in k^2

$$(Gr_1(k^2))^4 = \mathbb{P}^1(k)^4$$

and the group

$$GL(F(\xi_0)) \simeq GL(k^2)$$

acts on $\mathbb{P}^1(k)^4$ componentwise.
Look at the following invariant subspace $\Omega \subseteq \mathbb{P}^1(k)^4$

$$\Omega := \left\{(l_1, l_2, l_3, l_4) \in \mathbb{P}^1(k)^4 : l_i \neq l_j, \forall i \neq j\right\}$$

Action of $GL_k(k^2)$ on Ω can simultaneously transfer

$$l_1 \to \text{x-axis},\ l_2 \to \text{y-axis},\ l_3 \to \text{diagonal line}$$

Parametrization: Non-existence of complete discrete invariants

Example (cont.)

We have

$$\begin{aligned}\Omega/GL(k^2) &\Leftrightarrow \{\text{Lines in } k^2 \text{ with axes and diagonal removed}\}\\ &\Leftrightarrow k - \{0, 1\}.\end{aligned}$$

Parametrization: Non-existence of complete discrete invariants

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We have

$$\begin{aligned}\Omega/GL(k^2) &\Leftrightarrow \{\text{Lines in } k^2 \text{ with axes and diagonal removed}\}\\ &\Leftrightarrow k - \{0,1\}.\end{aligned}$$

Implication: non-existence of complete discrete invariants

If k is uncountable, then a complete parametrization has to be uncountable.
If k is finite, then orbit space is finite. But still not done yet!
So a discrete(countable) parametrization does not exist.

What can we do?

(1) There are still useful invariants that can give us information of the data set. For example, the rank invariant proposed in [Carlsson and A. Zomorodian 2009].

(2) Stability and finiteness of the rank invariant: Cerri et al. 2013 and [Cagliari and Landi 2011].

(3) Maybe the complete invariants are not necessary as long as we can extract enough useful information? [Scolamiero et al. 2017].

(4) Computation in multidimensional persistence. [Carlsson, Singh, and A. J. Zomorodian 2010].

Cagliari, Francesca and Claudia Landi (2011). "Finiteness of rank invariants of multidimensional persistent homology groups". In: *Applied Mathematics Letters* 24.4, pp. 516–518.

Carlsson, Gunnar, Gurjeet Singh, and Afra J Zomorodian (2010). "Computing multidimensional persistence". In: *Journal of Computational Geometry* 1.1, pp. 72–100.

Carlsson, Gunnar and Afra Zomorodian (2009). "The theory of multidimensional persistence". In: *Discrete & Computational Geometry* 42.1, pp. 71–93.

Carlsson, Gunnar, Afra Zomorodian, et al. (2005). "Persistence barcodes for shapes". In: *International Journal of Shape Modeling* 11.02, pp. 149–187.

Cerri, Andrea et al. (2013). "Betti numbers in multidimensional persistent homology are stable functions". In: *Mathematical Methods in the Applied Sciences* 36.12, pp. 1543–1557.

McInnes, L. and J. Healy (2017). "Accelerated Hierarchical Density Based Clustering". In: *2017 IEEE International Conference on Data Mining Workshops (ICDMW)*, pp. 33–42.

Scolamiero, Martina et al. (2017). “Multidimensional persistence and noise”. In: *Foundations of Computational Mathematics* 17.6, pp. 1367–1406.