Localization in HoTT

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Outline:

- Motivation for localization
- Main results about p-localization
- Proofs and background results

Motivation for localization

Localization of spaces was developed by Adams, Bousfield, Dror, Mimura, Nishida, Quillen, Sullivan, Toda, etc., starting in the 1970s.

It is now a fundamental and pervasive tool in algebraic topology.

There are many important theorems whose statement does not involve localization but which can be proved using localization. E.g.

Theorem (Serre). If Y is a simply connected, finite CW complex then either:

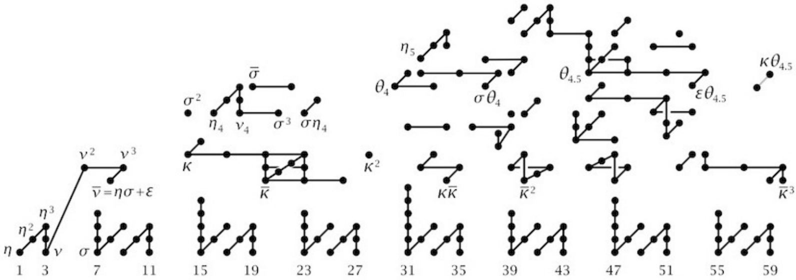
- Y is contractible, or
- $\pi_i Y$ is non-zero for infinitely many i.

Motivation for localization II

On the other hand, some theorems can only be stated using localization.

For example, there are patterns in the homotopy groups of spheres for which the periodicity in the pattern is different for summands whose torsion involves different primes. Image credit: Hatcher.

$p = 2:$

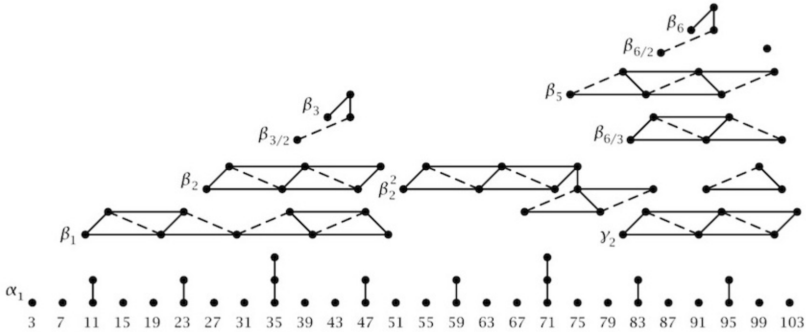


Motivation for localization II

On the other hand, some theorems can only be stated using localization.

For example, there are patterns in the homotopy groups of spheres for which the periodicity in the pattern is different for summands whose torsion involves different primes. Image credit: Hatcher.

$p = 3:$

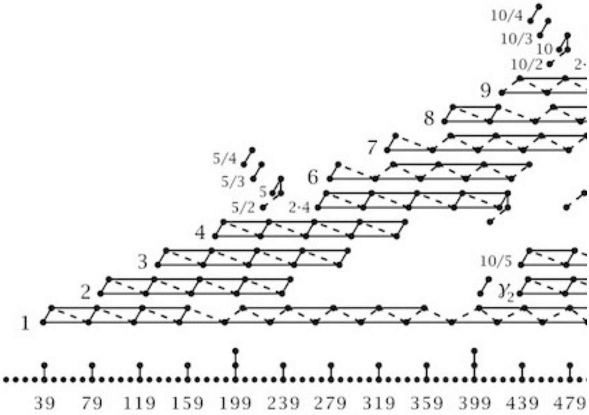


Motivation for localization II

On the other hand, some theorems can only be stated using localization.

For example, there are patterns in the homotopy groups of spheres for which the periodicity in the pattern is different for summands whose torsion involves different primes. Image credit: Hatcher.

$p = 5:$



Motivation for localization II

On the other hand, some theorems can only be stated using localization.

For example, there are patterns in the homotopy groups of spheres for which the periodicity in the pattern is different for summands whose torsion involves different primes. Image credit: Hatcher.

To study such phenonmena, it's useful to replace the sphere with a "p-localized" version which only contains the p-primary part of the homotopy groups.

Many papers in algebraic topology start with the phrase "In this paper, we are working localized at a prime p" and then implicitly invoke localization technology throughout.

Many computational techniques, such as the Adams spectral sequence, also work one prime at a time.

Motivation for localization III

A special case of localization is rationalization, which has the effect of tensoring all homotopy groups with $\mathbb{Q}$.

It turns out that the homotopy theory of rational spaces can be described completely algebraically (Quillen, Sullivan).
The algebraic description is very practical for computations.

Using rationalization, one can prove:

Theorem (Serre). The groups $\pi_i(S^n)$ are all finite, except $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{finite}$.

Localization is also a powerful tool for constructing counterexamples.

The work I'll describe brings localization into type theory, which is a necessary first step towards the results mentioned above.

p-Local types

I'm working in Book HoTT for the rest of the talk.
Fix a prime $p : \mathbb{N}$.

Def. A type X is p-local if for every prime $q \neq p$ and every $x_0 : X$, the map

$$q : \Omega(X, x_0) \longrightarrow \Omega(X, x_0) \quad \text{sending} \quad \ell \longmapsto \ell^q$$

is an equivalence.

Prop. The p-local types are closed under products, pullbacks, identity types and dependent products indexed by any type.
The unit type is p-local.

Def. A p-localization of X is a p-local type $X_{(p)}$ and a map $\eta : X \to X_{(p)}$ such that for every p-local type Z, every map $X \to Z$ factors uniquely through $X \to X_{(p)}$.

Theorem (Rijke, Shulman, Spitters). Every type X has a p-localization, unique up to equivalence, and functorial.

Main results

Theorem (CORS). For X simply connected, the natural map $\pi_n(X, x_0) \to \pi_n(X_{(p)}, \eta(x_0))$ is p-localization of abelian groups for every $n : \mathbb{N}$ and every $x_0 : X$.

The converse holds when X is truncated.

Theorem (Scoccola). Let R and S be denumerable sets of primes such that $R \cup S =$ all primes. Then, for X simply connected,

$$\begin{array}{ccc} X & \longrightarrow & X_{(R)} \\ \downarrow & & \downarrow \\ X_{(S)} & \longrightarrow & X_{(R \cap S)} \end{array}$$

is a pullback square.

Scoccola has also developed the theory of nilpotent types, which can have non-trivial fundamental group, and has generalized the above results to such types. (For the second theorem, he needs to assume that X is truncated in this case.)

Proof outline

Goal. $\pi_n(X) \to \pi_n(X_{(p)})$ is p-localization.

Prop 1 (CORS). For simply connected types, p-localization and n-truncation commute. [Proof later.]

In particular, if X is simply connected, then so is $X_{(p)}$.
The case $n = 1$ follows.

Case $n > 1$: Consider the fiber sequence

$$K(\pi_{n+1}(X), n+1) \longrightarrow \|X\|_{n+1} \longrightarrow \|X\|_n.$$

Applying p-localization gives

$$K(\pi_{n+1}(X), n+1)_{(p)} \longrightarrow \|X_{(p)}\|_{n+1} \longrightarrow \|X_{(p)}\|_n,$$

where we have used Prop 1 again.

We'll show that this is again a fibre sequence and that the fibre is $K(\pi_{n+1}(X)_{(p)}, n+1)$, which will complete the proof.

p-Separated types

p-localization is not lex, i.e., it does not preserve all fibre sequences. To work around this, we introduce:

Def. A type X is p-separated if for every $x, y : X$, the type $x = y$ is p-local.

Theorem (RSS). Every type X has a universal map $\eta' : X \to X'_{(p)}$ to a p-separated type.

We prove:

Theorem (CORS). Any fibre sequence fits into a diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & X \\ {\scriptstyle p\text{-equiv}}\downarrow & & \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta'} \\ F' & \longrightarrow & E'_{(p)} & \longrightarrow & X'_{(p)}, \end{array}$$

where F' is the fibre of the bottom row and is therefore p-separated.

Proof, continued

Then we use:

Prop 2 (CORS). For X simply connected, $X_{(p)} \simeq X'_{(p)}$.

Since the types in

$$K(\pi_{n+1}(X), n+1) \longrightarrow \|X\|_{n+1} \longrightarrow \|X\|_n$$

are all simply connected, we get a fibre sequence

$$F' \longrightarrow \|X_{(p)}\|_{n+1} \longrightarrow \|X_{(p)}\|_n,$$

and a p-equivalence $K(\pi_{n+1}(X), n+1) \to F'$.

Since F' is p-local, this map must be p-localization.

(More generally, p-localization preserves fibre sequences of simply connected types. So, for X pointed and simply connected, $\Omega(X_{(p)}) \simeq (\Omega X)_{(p)}$.)

It remains to understand the p-localization of an Eilenberg-Mac Lane space.

Localizations of Eilenberg-Mac Lane spaces

Prop (CORS). For X pointed and simply connected, the natural map

$$\Omega X \longrightarrow \mathrm{colim}(\Omega X \xrightarrow{k_1} \Omega X \xrightarrow{k_2} \cdots)$$

is the p-localization of ΩX, where k_i is the product of the first i primes, excluding p.

Proof. It's not too hard to see that the map is a p-equivalence.

To see that it is p-local uses the compactness of S^1, which uses the work of van Doorn, Rijke and Sojakova on the identity types of sequential colimits. □

Cor (CORS). For G abelian and $n \geq 1$, the p-localization of $K(G, n)$ is $K(G_{(p)}, n)$, where $G_{(p)}$ is the p-localization of G as an abelian group.

It follows that $\pi_n(X) \to \pi_n(X_{(p)})$ is p-localization.

Localization and truncation commute

We used:

Prop 1. For simply connected types, p-localization and n-truncation commute.

This follows from:

Lemma 1. The n-truncation of a p-local type is p-local.

Lemma 2. The p-localization of a simply connected n-truncated type is n-truncated.

Indeed, the natural maps

$$X \to X_{(p)} \to \left\|X_{(p)}\right\|_n$$

and

$$X \to \left\|X\right\|_n \to (\left\|X\right\|_n)_{(p)}$$

are both universal maps to types that are both n-truncated and p-local.

Proof of Lemma 1

Lemma 1. The n-truncation of a p-local type X is p-local.

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} \Omega\|X\|_n & \xleftarrow{\sim} & \|\Omega X\|_{n-1} \\ {\scriptstyle q}\downarrow & & \sim\downarrow{\scriptstyle \|q\|_{n-1}} \\ \Omega\|X\|_n & \xleftarrow{\sim} & \|\Omega X\|_{n-1}. \end{array}$$

where q is a prime different from p. □

Proof of Lemma 2

Lemma 2. The p-localization of a simply connected n-truncated type X is n-truncated.

Proof. By induction on n.

Trivial when $n \leq 1$, so assume X is simply connected and $(n+1)$-truncated for $n > 0$. Consider the fibre sequence

$$K(\pi_{n+1}(X), n+1) \longrightarrow X \longrightarrow \left\|X\right\|_n.$$

These are all simply connected, so by an earlier Theorem and Prop 2, we get another fibre sequence

$$K(\pi_{n+1}(X), n+1)_{(p)} \longrightarrow X_{(p)} \longrightarrow (\left\|X\right\|_n)_{(p)}.$$

The fibre and base are $(n+1)$-truncated (using the Cor about EM spaces), and so $X_{(p)}$ is $(n+1)$-truncated as well. □

References

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