# The Hurewicz Theorem in Homotopy Type Theory

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## Outline:

- Statement
- The ingredients
- Applications
- Key result and techniques

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# The Hurewicz theorem

**Thm H.** For $X$ a pointed, $(n-1)$-connected type $(n \geq 1)$, and $A$ an abelian group, there is a natural isomorphism

$$\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X; A).$$

On the left-hand-side we take the abelianization (which only matters when $n = 1$). In particular, when $A$ is the integers, this specializes to an isomorphism $\pi_n(X)^{ab} \cong \tilde{H}_n(X)$.

## Ingredients

- The $n$th homotopy group $\pi_n(X)$ of a pointed type $X$ is $\left\|\Omega^n X\right\|_0$, where $\Omega^n X$ denotes the iterated loop space. This is a set when $n = 0$ and a group when $n \geq 1$, and is abelian when $n \geq 2$.
- A type $X$ is $(n-1)$-connected if $\pi_k(X)$ is contractible for all $k \leq n-1$ and all basepoints.
  Equivalently, if its $(n-1)$-truncation $\left\|X\right\|_{n-1}$ is contractible.

# The ingredients: Groups

**Thm H.** For $X$ $(n-1)$-connected, $\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X; A)$.

A group is a set (0-truncated type) with an associative, unital binary operation with inverses.

An abelianization of a group $G$ is a homomorphism $G \to_{\mathsf{Grp}} G^{ab}$ to an abelian group that is initial among such homomorphisms.

A tensor product of abelian groups $A$ and $B$ is an abelian group $A \otimes B$ together with a map $t : A \to_{\mathsf{Grp}} B \to_{\mathsf{Grp}} A \otimes B$ which is initial among such bilinear maps.

It remains to explain the right-hand side.

# The ingredients: Homology

The $n$th homology group $\tilde{H}_n(X; A)$ is defined to be the colimit of a certain sequential diagram

$$\pi_{n+1}(X \wedge K(A,1)) \to \pi_{n+2}(X \wedge K(A,2)) \to \pi_{n+3}(X \wedge K(A,3)) \to \cdots .$$

$X \wedge Y$ denotes the smash product, which has points $X \times Y$, $\mathsf{L}$ and $\mathsf{R}$, and paths $(x_0, y) = \mathsf{L}$ and $(x, y_0) = \mathsf{R}$ for $x : X$ and $y : Y$.

$K(A, m)$ is an Eilenberg–Mac Lane space, which is an $m$-truncated, $(m-1)$-connected, pointed type with $\pi_m(K(A,m)) \cong A$.

We write $\tilde{H}_n(X)$ for $\tilde{H}_n(X; \mathbb{Z})$.

Unfortunately, we know very little about homology in HoTT.

# Applications

**Thm H.** For $X$ $(n-1)$-connected, $\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X;A)$.

**Cor.** If $X$ is $(n-1)$-connected, then $\tilde{H}_i(X) = 0$ for all $i \leq n-1$.
If $X$ is 1-connected and $\tilde{H}_i(X) = 0$ for all $i \leq n-1$,
then $X$ is $(n-1)$-connected.

**Cor.** $\tilde{H}_n(K(A,n)) \cong A$ for every $n \geq 1$ and $A : \mathsf{Ab}$.

# Applications, II

**Thm H.** For $X$ $(n-1)$-connected, $\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X; A)$.

Can be used to compute $\pi_n(X)$ even if $X$ is not $(n-1)$-connected.

The $(n-1)$-connected cover $X\langle n-1\rangle$ is the fibre of the truncation map $X \to \|X\|_{n-1}$ over the image of the basepoint.

$X\langle n-1\rangle$ is $(n-1)$-connected and $\pi_n(X\langle n-1\rangle) \cong \pi_n(X)$, so

$$\pi_n(X)^{ab} \cong \tilde{H}_n(X\langle n-1\rangle).$$

The Serre spectral sequence can often be used to compute the required homology group.

# Key result

**Thm S.** If $X$ is a pointed, $(n-1)$-connected type $(n \geq 1)$ and $Y$ is a pointed, $(m-1)$-connected type $(m \geq 1)$, then $X \wedge Y$ is $(n+m-1)$-connected and

$$\pi_{n+m}(X \wedge Y) \cong \pi_n(X)^{ab} \otimes \pi_m(Y)^{ab}.$$

**Ex.** For $X$ $(n-1)$-connected,

$$\pi_{n+m}(X \wedge K(A,m)) \cong \pi_n(X)^{ab} \otimes A.$$

These are the groups used to define $\tilde{H}_n(X; A)$ as the colimit

$$\pi_{n+1}(X \wedge K(A,1)) \to \pi_{n+2}(X \wedge K(A,2)) \to \pi_{n+3}(X \wedge K(A,3)) \to \cdots .$$

The Hurewicz theorem follows from showing that the induced maps are isomorphisms.

# Techniques

To prove that $\pi_{n+m}(X \wedge Y) \cong \pi_n(X)^{ab} \otimes \pi_m(Y)^{ab}$, we need to give a map $\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)$.

We define and study a more general map

$$\mathsf{smashing} : (X \to_\bullet Y \to_\bullet Z) \longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z))$$

for any pointed types $X$, $Y$ and $Z$ and any $n, m \geq 1$.

The bilinear map we require is obtained by applying smashing to the natural map $X \to_\bullet Y \to_\bullet X \wedge Y$.

In topology, smashing $f$ is defined by taking representatives $S^n \to_\bullet X$ and $S^m \to_\bullet Y$, *smashing* them together to get $S^{n+m} \simeq S^n \wedge S^m \to_\bullet X \wedge Y \xrightarrow{\bar{f}}_\bullet Z$, where $\bar{f}$ is the adjoint of $f$.

However, we define it in a way that makes it easier to prove that it has the properties that we need:

# A smashing idea

The map $\mathsf{smashing}$ is defined as a composite:

$$\begin{aligned}
(X \to_\bullet Y \to_\bullet Z) &\longrightarrow (\Omega^n X \to_{\mathsf{Mgm}} \Omega^n (Y \to_\bullet Z)) \\
&\xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}} (Y \to_\bullet \Omega^n Z)) \\
&\longrightarrow (\Omega^n X \to_{\mathsf{Mgm}} (\Omega^m Y \to_{\mathsf{Mgm}} \Omega^m \Omega^n Z)) \\
&\xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}} (\Omega^m Y \to_{\mathsf{Mgm}} \Omega^{n+m} Z)) \\
&\longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z)).
\end{aligned}$$

However, we'd need to land in group homomorphisms, so we pass through magmas.

A magma is a type with a binary operation, and a magma morphism is a map that respects the operations.

We define the type $M \to_{\mathsf{Mgm}} N$ of weak magma morphisms to be the type of maps $M \to N$ that merely respect the operations.

# Further results

We also prove the following results:

**Prop.** Let $X$ be a pointed type and let $n \leq m$ be natural numbers. Then the truncation map $X \to \|X\|_m$ induces an isomorphism

$$\tilde{H}_n(X; A) \xrightarrow{\sim} \tilde{H}_n\left(\|X\|_m; A\right).$$

**Cor.** Let $f : X \to_\bullet Y$ be a pointed map that induces an isomorphism in $\pi_0$ and an isomorphism in $\pi_n$ for $n \geq 1$ and all basepoints. Then $f$ induces an isomorphism in all homology groups.

# Final words

Formalization of this work is in progress using the HoTT Coq library, with help from Ali Caglayan.

These slides and a preprint are on my web site:

http://jdc.math.uwo.ca/h

**Thanks for listening!**