DIFFEOLOGICAL VECTOR SPACES

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Abstract. We study the relationship between many natural conditions that one can put on a diffeological vector space, including being fine or projective, having enough smooth (or smooth linear) functionals to separate points, having a diffeology determined by the smooth linear functionals, and more. Our main result is that the majority of the conditions fit into a total order. We also give many examples in order to show which implications do not hold, and use our results to study the homological algebra of diffeological vector spaces.

Contents

1. Introduction

Diffeological spaces are elegant generalizations of manifolds that include a variety of singular spaces and infinite-dimensional spaces. Many vector spaces that arise in applications are naturally equipped with a compatible structure of a diffeological space. Examples include $C^\infty(M, \mathbb{R}^n)$ for a manifold (or even a diffeological space) M, spaces of smooth or holomorphic sections of vector bundles, tangent spaces of diffeological spaces (as defined in [CW]), smooth duals of all of these spaces, etc. Such objects are called diffeological vector spaces and are the topic of this paper.

Diffeological vector spaces have been studied by Iglesias-Zemmour in [I1, I2]. He used them to define diffeological manifolds, and developed the theory of *fine* diffeological vector spaces, a particularly well-behaved kind that forms the beginning of our story. In [KM], Kriegl and Michor studied topological vector spaces equiped with a smooth structure, and their examples can be regarded as diffeological vector spaces. Diffeological vector spaces were used in the study of tangent spaces of diffeological spaces in [V] and [CW]. In their work on factorization algebras in quantum field theory [CG], Costello and Gwilliam studied a class of diffeological vector spaces that they called differentiable vector spaces, and developed their homological algebra in detail. Wu investigated the

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homological algebra of *all* diffeological vector spaces [Wu], and the present paper builds heavily on this foundation.

In this paper, we study some natural conditions that one can put on a diffeological vector space, and show that the majority of them fit into a total order. In order to state our results, we briefly introduce the conditions here, making use of some background material summarized in Section 2.

Any vector space has a smallest diffeology making it into a diffeological vector space. This is called the **fine** diffeology, and we write $\mathcal{FV}$ for the collection of vector spaces with the fine diffeology. We write $\mathcal{FFV}$ for the collection of diffeological vector spaces whose finite-dimensional subspaces (with the induced diffeology) are all fine.

A diffeological vector space V is **projective** if for every linear subduction $f : W_1 \to W_2$ and every smooth linear map $g : V \to W_2$, there exists a smooth linear map $h : V \to W_1$ such that $g = f \circ h$. We write $\mathcal{PV}$ for the collection of projective diffeological vector spaces.

A diffeological vector space V is in $\mathcal{SD}$ (resp. $\mathcal{SV}$) if the smooth (resp. smooth linear) functionals $V \to \mathbb{R}$ separate points of V. That is, for each x and y in V with $x \neq y$, such a functional f can be found so that $f(x) \neq f(y)$.

Each diffeological space has a natural topology called the D-topology. We write $\mathcal{HT}$ for the collection of diffeological vector spaces whose D-topologies are Hausdorff.

The last letter of the abbreviation is $\mathcal{V}$, $\mathcal{D}$ or $\mathcal{T}$ depending on whether the condition depends on the structure as a diffeological vector space, a diffeological space, or a topological space.

We now state the main results of the paper.

Theorem 1.1. *We have the following chain of containments:*

$$\mathcal{FV} \subset \mathcal{PV} \subset \mathcal{SV} \subseteq \mathcal{SD} \subseteq \mathcal{FFV} \quad \textit{and} \quad \mathcal{SD} \subset \mathcal{HT},$$

where $\subset$ indicates proper containment. We also know that $\mathcal{HT} \not\subseteq \mathcal{FFV}$, but do not know whether the reverse inclusion holds.

The property of being finite-dimensional does not imply, nor is it implied by, any of the properties considered above. However, under this assumption, most of the properties agree.

Theorem 1.2. *When restricted to finite-dimensional vector spaces, the collections $\mathcal{FV}, \mathcal{PV}, \mathcal{SV}, \mathcal{SD}$ and $\mathcal{FFV}$ agree.*

Indeed, $\mathcal{FV}$ and $\mathcal{FFV}$ clearly agree for finite-dimensional spaces, so the containments must collapse to equalities. Note that we prove part of Theorem 1.2 on the way to proving Theorem 1.1.

The final property we consider is the following. Write $\mathcal{DV}$ for the collection of diffeological vector spaces V such that a function $p : \mathbb{R}^n \to V$ is smooth if and only if $\ell \circ p : \mathbb{R}^n \to \mathbb{R}$ is smooth for each smooth linear functional $\ell : V \to \mathbb{R}$. Except for the inclusion $\mathcal{FV} \subset \mathcal{DV}$, the class $\mathcal{DV}$ is independent of all of the others we have considered. However, under this assumption, we again find that many of the other conditions agree.

Theorem 1.3. *When restricted to V in $\mathcal{DV}$, the collections $\mathcal{SV}, \mathcal{SD}, \mathcal{FFV}$ and $\mathcal{HT}$ agree.*

The proofs of the containments, and the examples showing that many inclusions do not hold, are spread throughout Section 3. The only exception is that the longest argument, which is the proof that $\mathcal{SD} \subseteq \mathcal{FFV}$, is deferred until Section 5. Along the way, we also prove other results, such as new characterizations of the fine diffeology, and some necessary conditions for diffeological vector spaces and free diffeological vector spaces to be projective. In Section 4, we give some applications of our results to the homological algebra of diffeological vector spaces.

We thank Chengjie Yu for the argument used in Case 1 of the proof of Theorem 3.20 in Section 5.

2. Background and conventions

In this section, we briefly recall some background on diffeological spaces. For further details, we recommend the standard textbook [I2]. For a concise introduction to diffeological spaces, we recommend [CSW], particularly Section 2 and the introduction to Section 3.

Definition 2.1 ([So]). *A **diffeological space** is a set X together with a specified set of functions $U \to X$ (called **plots**) for each open set U in $\mathbb{R}^n$ and each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$:*

1. *(Covering) Every constant function $U \to X$ is a plot.*
2. *(Smooth Compatibility) If $U \to X$ is a plot and $V \to U$ is smooth, then the composite $V \to U \to X$ is also a plot.*
3. *(Sheaf Condition) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a function such that each restriction $U_i \to X$ is a plot, then $U \to X$ is a plot.*

*A function $f : X \to Y$ between diffeological spaces is **smooth** if for every plot $p : U \to X$ of X, the composite $f \circ p$ is a plot of Y.*

The category of diffeological spaces and smooth maps is complete and cocomplete. Given two diffeological spaces X and Y, we write $C^\infty(X, Y)$ for the set of all smooth maps from X to Y. An isomorphism in Diff will be called a **diffeomorphism**.

Every manifold M is canonically a diffeological space with the plots taken to be all smooth maps $U \to M$ in the usual sense. We call this the **standard diffeology** on M. It is easy to see that smooth maps in the usual sense between manifolds coincide with smooth maps between them with the standard diffeology.

For a diffeological space X with an equivalence relation $\sim$, the **quotient diffeology** on $X/{\sim}$ consists of all functions $U \to X/{\sim}$ that locally factor through the quotient map $X \to X/{\sim}$ via plots of X. A **subduction** is a map diffeomorphic to a quotient map. That is, it is a map $X \to Y$ such that the plots in Y are the functions that locally lift to X as plots in X.

For a diffeological space Y and a subset A of Y, the **sub-diffeology** consists of all functions $U \to A$ such that $U \to A \hookrightarrow Y$ is a plot of Y. An **induction** is an injective smooth map $A \to Y$ such that a function $U \to A$ is a plot of A if and only if $U \to A \to Y$ is a plot of Y.

The **discrete diffeology** on a set is the diffeology whose plots are the locally constant functions. The **indiscrete diffeology** on a set is the diffeology in which every function is a plot.

We can associate to every diffeological space the following topology:

Definition 2.2 ([I1]). *Let X be a diffeological space. A subset A of X is **D-open** if $p^{-1}(A)$ is open in U for each plot $p : U \to X$. The collection of D-open subsets of X forms a topology on X called the **D-topology**.*

Definition 2.3. *A **diffeological vector space** is a vector space V with a diffeology such that addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ are smooth.*

Let V be a diffeological vector space. We write $L^\infty(V, \mathbb{R})$ for the set of all smooth linear maps $V \to \mathbb{R}$, and $L(V, \mathbb{R})$ for the set of all linear maps $V \to \mathbb{R}$. We write DVect for the category of diffeological vector spaces and smooth linear maps.

Conventions

Throughout this paper, we use the following conventions. Every subset of a diffeological space is equipped with the sub-diffeology and every product is equipped with the product diffeology. Every vector space is over the field $\mathbb{R}$ of real numbers, and every linear map is $\mathbb{R}$-linear. By a subspace of a diffeological vector space, we mean a linear subspace with the sub-diffeology. All manifolds are smooth, finite-dimensional, Hausdorff, second countable and without boundary, and are equipped with the standard diffeology.

3. Diffeological vector spaces

In this section, we study a variety of conditions that a diffeological vector space can satisfy. Together, the results described here give the theorems stated in the introduction. In addition, we present some auxiliary results, and give many examples and counterexamples.

3.1. **Fine diffeological vector spaces**

In this subsection, we recall background on the fine diffeology, and then give two new characterizations.

Given a vector space V, the set of all diffeologies on V each of which makes V into a diffeological vector space, ordered by inclusion, is a complete lattice. This follows from [CW, Proposition 4.6], taking X to be a point. The largest element in this lattice is the indiscrete diffeology, which is usually not interesting. Another extreme has the following special name in the literature:

Definition 3.1. *The* ***fine*** *diffeology on a vector space V is the smallest diffeology on V making it into a diffeological vector space.*

For example, the fine diffeology on $\mathbb{R}^n$ is the standard diffeology.

Remark 3.2. The fine diffeology is generated by the injective linear maps $\mathbb{R}^n \to V$ ([I2, 3.8]). That is, the plots of the fine diffeology are the functions $p : U \to V$ such that for each $u \in U$, there is an open neighbourhood W of u in U, an injective linear map $i : \mathbb{R}^n \to V$ for some $n \in \mathbb{N}$, and a smooth map $f : W \to \mathbb{R}^n$ such that $p|_W = i \circ f$.

One can show that if V is fine and a plot $W \to V$ factors set-theoretically through an injective linear map $\mathbb{R}^n \to V$, then the map $W \to \mathbb{R}^n$ is smooth. More generally, every subspace of a fine diffeological vector space is fine ([Wu]).

In fact, fineness of a diffeological vector space can be tested by smooth curves:

Proposition 3.3. *A diffeological vector space V is fine if and only if for every plot $p : \mathbb{R} \to V$ and every $x \in \mathbb{R}$, there exist an open neighborhood W of x in $\mathbb{R}$, an injective linear map $i : \mathbb{R}^n \to V$ for some $n \in \mathbb{N}$, and a smooth map $f : W \to \mathbb{R}^n$ such that $p|_W = i \circ f$.*

Proof. ($\Rightarrow$) This follows from the description of the fine diffeology in Remark 3.2.

($\Leftarrow$) Assume that V is not fine. Then, by the last part of Remark 3.2, there exist a plot $q : U \to V$ and a point $u \in U$ such that one of the following two conditions holds: (1) for every open neighborhood W of u in U, $q|_W$ does not land in any finite-dimensional subspace of V; (2) there exist an open neighborhood W of u in U, a linear injection $i : \mathbb{R}^n \to V$ and a non-smooth map $g : W \to \mathbb{R}^n$ such that $q|_W = i \circ g$. If (1) holds, then there exists a sequence u_i in U converging to u such that $\{q(u_i) \mid i \in \mathbb{Z}^+\}$ is linearly independent in V. We may assume that the sequence u_i converges fast to u (see [KM, I.2.8]). By the Special Curve Lemma [KM, I.2.8], there exists a smooth map $f : \mathbb{R} \to U$ such that $f(1/i) = u_i$ and $f(0) = u$. Then $q \circ f : \mathbb{R} \to V$ is a plot which does not satisfy the hypothesis at $x = 0$. If (2) holds, then by Boman's theorem (see, e.g., [KM, Corollary 3.14]), there exists a smooth curve $r : \mathbb{R} \to W$ such that $g \circ r$ is not smooth, which again conflicts with the hypothesis. □

Proposition 3.4. *A diffeological vector space V is fine if and only if $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$, i.e., if and only if every linear functional is smooth.*

Proof. This follows from the proof of [Wu, Proposition 5.7]. We give a direct proof here.

It is easy to check that if V is fine, then every linear functional is smooth.

To prove the converse, suppose that every linear functional $V \to \mathbb{R}$ is smooth. Let $p : U \to V$ be a plot and let $u \in U$. First we must show that when restricted to a neighbourhood of u, p lands in a finite-dimensional subspace of V. If not, then there is a sequence $\{u_j\}$ converging to u such that the vectors $p(u_j)$ are linearly independent. Thus there is a linear functional $l : V \to \mathbb{R}$ such that $p(u_j)$ is sent to 1 when j is odd and 0 when j is even. By assumption, l is smooth. But $l \circ p$ is not continuous, contradicting the fact that p is a plot.

So now we know that p locally factors through an injective linear map $i : \mathbb{R}^n \to V$. (Of course, n may depend on the neighborhood.) For each $1 \leq j \leq n$, there is a linear map $l_j : V \to \mathbb{R}$ such that $l_j \circ i$ is projection onto the j^{th} coordinate. Since $l_j \circ p$ is smooth, it follows that the local factorizations through $\mathbb{R}^n$ are smooth. Thus V is fine. □

3.2. Projective diffeological vector spaces

A diffeological vector space V is **projective** if for every linear subduction $f : W_1 \to W_2$ and every smooth linear map $g : V \to W_2$, there exists a smooth linear map $h : V \to W_1$ making the diagram

$$\begin{tikzcd} & W_1 \arrow[d, "f"] \\ V \arrow[ur, dashed, "h"] \arrow[r, "g"'] & W_2 \end{tikzcd}$$

commute. We write $\mathcal{PV}$ for the collection of projective diffeological vector spaces.

Example 3.5. Write $F(X)$ for the free diffeological vector space generated by a diffeological space X (see [Wu, Proposition 3.5]). This has a basis consisting of the elements of X, and has the smallest diffeology making it into a diffeological vector space and such that the natural map $X \to F(X)$ is smooth. By [Wu, Corollary 6.4], when M is a manifold, $F(M)$ is projective. However, by [Wu, Theorem 5.3], $F(X)$ is fine if and only if X is discrete. So not every projective diffeological vector space is fine.

Proposition 3.6 ([Wu, Corollary 6.3])**.** *Every fine diffeological vector space is projective.*

Proof. This follows immediately from Proposition 3.4. One can take h to be $k \circ g$, where k is a linear section of f (which is not necessarily smooth). □

Projective diffeological vector spaces and the homological algebra of diffeological vector spaces are studied further in [Wu].

3.3. Separation of points

Definition 3.7. *Let X be a diffeological space. A set A of functions with domain X is said to **separate points** if for any $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$. We say that the **smooth functionals separate points** if $C^\infty(X, \mathbb{R})$ separates points. We write $\mathcal{SD}'$ for the collection of all such diffeological spaces X and $\mathcal{SD}$ for the diffeological vector spaces whose underlying diffeological spaces are in $\mathcal{SD}'$. If V is a diffeological vector space, we say that the **smooth linear functionals separate points** if $L^\infty(V, \mathbb{R})$ separates points, and we write $\mathcal{SV}$ for the collection of all such diffeological vector spaces V.*

We will establish basic properties of such diffeological vector spaces below, and show that many familiar diffeological vector spaces have this property. Clearly, $\mathcal{SV} \subseteq \mathcal{SD}$.

Example 3.8. Every fine diffeological vector space is in $\mathcal{SV}$, since the coordinate functions with respect to any basis are smooth and linear. Every manifold is in $\mathcal{SD}'$, since charts provide smooth coordinates that distinguish points (or using Whitney's embedding theorem).

Proposition 3.9.

(1) *If $W \to V$ is a smooth linear injective map between diffeological vector spaces and $V \in \mathcal{SV}$, then $W \in \mathcal{SV}$. In particular, $\mathcal{SV}$ is closed under taking subspaces.*
(2) *Let $\{V_i\}_{i\in I}$ be a set of diffeological vector spaces. Then $\prod_{i\in I} V_i \in \mathcal{SV}$ if and only if each $V_i \in \mathcal{SV}$.*
(3) *Let $\{V_i\}_{i\in I}$ be a set of diffeological vector spaces. Then $\oplus_{i\in I} V_i \in \mathcal{SV}$ if and only if each $V_i \in \mathcal{SV}$, where $\oplus_{i\in I} V_i$ is the coproduct in* DVect *(see* [Wu, Proposition 3.2]*).*

Proof. This is straightforward. □

Proposition 3.10. *If $V \in \mathcal{SV}$ and X is a diffeological space, then $C^\infty(X, V) \in \mathcal{SV}$.*

Proof. This follows from the fact that every evaluation map $C^\infty(X, V) \to V$ is smooth and linear. □

Proposition 3.11. *The following are equivalent:*

(1) $X \in \mathcal{SD}'$.
(2) $F(X) \in \mathcal{SV}$.
(3) $F(X) \in \mathcal{SD}$.

Proof. It is enough to prove (1) $\Rightarrow$ (2), since (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are straightforward. Note that every element of $F(X)$ is a linear combination of finitely many elements of X. Since $C^\infty(X,\mathbb{R})$ separates points of X, for any $x, x_1, \ldots, x_n \in X$, there exists $f \in C^\infty(X,\mathbb{R})$ such that $f(x) = 1$ and $f(x_i) = 0$ for each i. The result then follows easily from the universal property of $F(X)$. $\square$

Proposition 3.12. *Every projective diffeological vector space is in $\mathcal{SV}$.*

Proof. By Example 3.8, every open subset U of a Euclidean space is in $\mathcal{SD}'$. So Proposition 3.11 implies that $F(U)$ is in $\mathcal{SV}$. Corollary 6.15 of [Wu] says that every projective diffeological vector space is a retract of a coproduct of $F(U)$'s in DVect. Therefore, it follows from Proposition 3.9(3) and (1) that every projective diffeological vector space is in $\mathcal{SV}$. $\square$

Remark 3.13.

(1) Not every diffeological vector space in $\mathcal{SV}$ is projective. For example, let $V := \prod_\omega \mathbb{R}$ be the product of countably many copies of $\mathbb{R}$. By Proposition 3.9(2), V is in $\mathcal{SV}$. But [Wu, Example 4.3] shows that V is not projective.
(2) $\mathcal{SV}$ is not closed under taking quotients in DVect. For example, $F(\pi) : F(\mathbb{R}) \to F(T_\alpha)$ is a linear subduction, where α is an irrational and $\pi : \mathbb{R} \to T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ is the projection to the 1-dimensional irrational torus. By Proposition 3.11, $F(\mathbb{R})$ is in $\mathcal{SV}$, but $F(T_\alpha)$ is not in $\mathcal{SV}$ since T_α is not in $\mathcal{SD}'$. In particular, the free diffeological vector space $F(T_\alpha)$ is not projective, as observed in [Wu, Example 4.3].

Here is an easy fact:

Proposition 3.14. *The D-topology of every diffeological space in $\mathcal{SD}'$ is Hausdorff. In particular, $\mathcal{SD} \subseteq \mathcal{HT}$.*

Proof. This follows from the fact that every smooth map is continuous when both domain and codomain are equipped with the D-topology. $\square$

Corollary 3.15. *If $F(X)$ is projective, then X is Hausdorff.*

Proof. If $F(X)$ is projective, then it is in $\mathcal{SD}$ by Proposition 3.12, and so X is in $\mathcal{SD}'$ by Proposition 3.11. Thus X is Hausdorff, by Proposition 3.14. $\square$

This gives another proof that free diffeological vector spaces are not always projective. For example, if a set X with more than one point is equipped with the indiscrete diffeology, then the D-topology on X is indiscrete as well, and hence $F(X)$ is not projective.

Example 3.16. The converse of Proposition 3.14 does not hold. Write $C(\mathbb{R})$ for the vector space $\mathbb{R}$ equipped with the continuous diffeology, so that a function $p : U \to V$ is a plot if and only if it is continuous ([CSW, Section 3]). Then $C(\mathbb{R})$ is a Hausdorff diffeological vector space, as the D-topology on $C(\mathbb{R})$ is the usual topology. But one can show that $C^\infty(C(\mathbb{R}),\mathbb{R})$ consists of constant functions ([CW, Example 3.15]), so $C(\mathbb{R})$ is not in $\mathcal{SD}$.

We will use the following result in the next subsection.

Theorem 3.17. *Let V be a finite-dimensional diffeological vector space. Then the following are equivalent:*

(1) *V is fine.*
(2) *V is projective.*
(3) *V is in $\mathcal{SV}$.*

Proof. By Propositions 3.6 and 3.12, (1) $\implies$ (2) $\implies$ (3), for all V. So it remains to prove (3) $\implies$ (1). Assume that V is finite-dimensional and in $\mathcal{SV}$. Choose a basis $f_1, \ldots, f_k$ for $L^\infty(V, \mathbb{R})$, and use it to give a smooth linear map $f : V \to \mathbb{R}^k$. Note that $k \leq \dim V$. Since V is in $\mathcal{SV}$, f is injective, and hence surjective. The diffeology on $\mathbb{R}^k$ is the fine diffeology, which is the smallest diffeology making it into a diffeological vector space. The map $f : V \to \mathbb{R}^k$ is a smooth linear bijection, so the diffeology on V must be fine as well (and f must be a diffeomorphism). □

The above result also follows from Proposition 3.4, since V in $\mathcal{SV}$ implies that $\dim L(V, \mathbb{R}) \geq \dim L^\infty(V, \mathbb{R}) \geq \dim V = \dim L(V, \mathbb{R})$, and so $L^\infty(V, \mathbb{R}) = L(V, \mathbb{R})$.

3.4. **Diffeological vector spaces whose finite-dimensional subspaces are fine**

Write $\mathcal{FFV}$ for the collection of diffeological vector spaces whose finite-dimensional subspaces are fine. One motiviation for studying this collection is the following. In [CW], we defined a diffeology on Hector's tangent spaces [He] which makes them into diffeological vector spaces. While they are not fine in general, we know of no examples that are not in $\mathcal{FFV}$.

As an example, one can show that $\prod_\omega \mathbb{R}$ is in $\mathcal{FFV}$. This also follows from the next result, which is based on a suggestion of Y. Karshon.

Theorem 3.18. *Every diffeological vector space in $\mathcal{SV}$ is in $\mathcal{FFV}$.*

This result is a special case of Theorem 3.20 below, but we provide a direct proof, since it follows easily from earlier results.

Proof. Let W be a finite-dimensional subspace of V with $V \in \mathcal{SV}$. By Proposition 3.9(1), W is in $\mathcal{SV}$, and then by Theorem 3.17, W is fine. □

Remark 3.19.

(1) Note that it is not in general true that every diffeological vector space in $\mathcal{SV}$ is fine. For example, $\prod_\omega \mathbb{R}$ is in $\mathcal{SV}$ by Remark 3.13(1), but it is not fine. In fact, [Wu, Example 5.4] showed that there is a countable-dimensional subspace of $\prod_\omega \mathbb{R}$ which is not fine. Incidentally, it follows that $\prod_\omega \mathbb{R}$ is not the colimit in DVect of its finite-dimensional subspaces, since fine diffeological vector spaces are closed under colimits ([Wu, Property 6 after Definition 5.2]).

(2) We don't know whether every diffeological vector space in $\mathcal{FFV}$ is Hausdorff. However, when $\mathbb{R}$ is equipped with the continuous diffeology (see Example 3.16), it is Hausdorff but is not in $\mathcal{FFV}$.

The main result of this section is the following:

Theorem 3.20. *Every diffeological vector space in $\mathcal{SD}$ is in $\mathcal{FFV}$.*

We defer the proof to Section 5.

Next we observe that if V is projective (and hence in $\mathcal{SV}$ and $\mathcal{SD}$), it does not follow that all countable-dimensional subspaces of V are fine. We will illustrate this with $V = F(\mathbb{R})$. By [Wu, Corollary 6.4], $F(\mathbb{R})$ is projective.

Proposition 3.21. *Let A be a subset of $\mathbb{R}$, and let V be the subspace of $F(\mathbb{R})$ spanned by A. Then V is fine if and only if A has no accumulation point in $\mathbb{R}$.*

For example, $F(\mathbb{R})$ is not fine. As a more interesting example, $A = \{1/n \mid n = 1, 2, \ldots\}$ spans a countable-dimensional subspace V of $F(\mathbb{R})$ which is not fine. It will follow from Proposition 3.22 that V is not free on any diffeological space.

Proof. ($\Leftarrow$) Let $p : U \to V$ be a plot, where U is open in some $\mathbb{R}^n$. Since V is the span of A, there exist unique functions $h_a : \mathbb{R} \to \mathbb{R}$ such that $p(x) = \sum_{a \in A} h_a(x)[a]$. Since A has no accumulation point in $\mathbb{R}$, for each a in A there exists a smooth bump function $\phi_a : \mathbb{R} \to \mathbb{R}$ which takes the value 1 at a and 0 at every other element of A. Associated to ϕ_a is a smooth linear map $\tilde{\phi}_a : F(\mathbb{R}) \to \mathbb{R}$

which sends $[a]$ to 1 and all other basis elements from A to 0. Then $h_a = \tilde{\phi}_a \circ p$, which shows that h_a is smooth.

Next we show that locally p factors through the span of a finite subset of A. Fix $u \in U$. As V is a subspace of $F(\mathbb{R})$, there is an open neighbourhood U' of u in U such that $p(x) = \sum_{j=1}^m f_j(x)[g_j(x)]$ for $x \in U'$, where f_j and g_j are smooth functions $U' \to \mathbb{R}$. Shrinking U' if necessary, we can assume that it is contained in a compact subset of U. It follows that the image of each g_j is contained in a compact subset of $\mathbb{R}$ and therefore intersects only finitely many points of A. Since there are only finitely many g_j's, $p|_{U'}$ factors through the span of A' for some finite subset A' of A. That is, $h_a(x) = 0$ for all $x \in U'$ and all $a \in A \setminus A'$.

In summary, identifying the span of A' with $\mathbb{R}^{A'}$, we have factored $p|_{U'}$ as $U' \to \mathbb{R}^{A'} \to V$, where the first map is $x \mapsto (h_a(x))_{a \in A'}$ and the second map sends $f : A' \to \mathbb{R}$ to $\sum_{a \in A'} f(a)[a]$.

($\Rightarrow$) Now we prove that if A has an accumulation point a_0 in $\mathbb{R}$, then V is not fine. Pick a sequence (a_i) in $A \setminus \{a_0\}$ that converges fast to a_0. Choose a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \neq 0$ for $1/(2n+1) < x < 1/2n$ for each $n \in \mathbb{Z}^+$, and $f(x) = 0$ for all other x. Choose another smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = a_n$ for $1/(2n+1) < x < 1/2n$ for each $n \in \mathbb{Z}^+$, with no constraints on g otherwise. It will necessarily be the case that $g(0) = a_0$, and such a smooth g exists because the sequence was chosen to converge fast. Then the function $p : \mathbb{R} \to V$ defined by $p(x) = f(x)[g(x)]$ is smooth, but there is no open neighbourhood U of 0 so that $p|_U$ factors through a finite-dimensional subspace of V. □

On the other hand, we have:

Proposition 3.22. *Let X be a diffeological space with a finite or countable underlying set. Then the following are equivalent:*

1. *X is discrete.*
2. *$F(X)$ is fine.*
3. *$F(X)$ is projective.*
4. *$F(X)$ is in $\mathcal{SV}$.*
5. *$F(X)$ is in $\mathcal{SD}$.*
6. *$F(X)$ is Hausdorff.*

Proof. That (1) $\implies$ (2) is straightforward. The implications (2) $\implies$ (3) $\implies$ (4) and (5) $\implies$ (6) follow from Propositions 3.6, 3.12 and 3.14, while (4) $\implies$ (5) is clear. None of these use the assumption on the cardinality of X.

It remains to prove that (6) $\implies$ (1). Since the natural injective map $X \to F(X)$ is smooth, it is also continuous when X and $F(X)$ are both equipped with the D-topology. Therefore, X is Hausdorff. We must show that the diffeology on X is discrete. Let $p : U \to X$ be a plot from a connected open subset U of a Euclidean space. We will show that p is constant. If not, then the image of p contains two distinct points $x, x' \in X$ which are connected by a continuous path $q : [0,1] \to X$. The image of q is compact Hausdorff, and therefore normal. So by Urysohn's lemma, there is a continous map $l : \mathrm{Im}(q) \to \mathbb{R}$ which separates x and x'. Hence, the composite $l \circ q : [0,1] \to \mathrm{Im}(q) \to \mathbb{R}$ has uncountable image, which is a contradiction, since $\mathrm{Im}(q) \subseteq X$ is at most countable. □

Part of the above proof is based on the argument in [Ha]. The implication (2) $\implies$ (1) is also proved in [Wu, Theorem 5.3], without assuming that X is finite or countable.

3.5. Diffeologies determined by smooth linear functionals

Definition 3.23. *The diffeology on a diffeological vector space V is **determined by its smooth linear functionals** if $p : U \to V$ is a plot if and only if $l \circ p$ is smooth for every $l \in L^\infty(V, \mathbb{R})$. Write $\mathcal{DV}$ for the collection of all such diffeological vector spaces.*

Note that any vector space with the indiscrete diffeology is in $\mathcal{DV}$. It follows that being in $\mathcal{DV}$ does not imply any of the other conditions we have studied.

Also note that every diffeological vector space V in $\mathcal{DV}$ is **Frölicher**: $p : U \to V$ is a plot if and only $f \circ p$ is smooth for every $f \in C^\infty(V, \mathbb{R})$. We do not know if the converse holds.

We will see that for diffeological vector spaces in $\mathcal{DV}$, the converse of Theorem 3.18 holds. For this, we need the following results.

Lemma 3.24.

(1) *If V is in $\mathcal{DV}$ and W is a subspace of V, then W is in $\mathcal{DV}$.*
(2) *Let $\{V_i\}$ be a set of diffeological vector spaces. Then each V_i is in $\mathcal{DV}$ if and only if $\prod V_i$ is in $\mathcal{DV}$.*

Since the category DVect is an additive category, (2) also implies that $\mathcal{DV}$ is closed under taking finite direct sums.

Proof. This is straightforward. □

Proposition 3.25. *Let V be a diffeological vector space. Then V is in $\mathcal{DV}$ if and only if V can be written as a direct sum $V \cong W_0 \oplus W_1$ of diffeological vector spaces, where W_0 is indiscrete and W_1 is in $\mathcal{SV} \cap \mathcal{DV}$.*

Proof. Given V in $\mathcal{DV}$, let W_0 be $\cap_{l \in L^\infty(V,\mathbb{R})} \ker(l)$ with the sub-diffeology. Since $L^\infty(V, \mathbb{R})$ determines the diffeology on V, W_0 is indiscrete. Let W_1 be the quotient V/W_0, with the quotient diffeology. If $v + W_0$ is a non-zero element of V/W_0, then $v \notin W_0$, so there is a smooth linear functional $l : V \to \mathbb{R}$ such that $l(v) \neq 0$. This l factors through V/W_0, so it follows that V/W_0 is in $\mathcal{SV}$. By the next lemma, we have $V \cong W_0 \oplus V/W_0$ as diffeological vector spaces.

The converse follows from the previous lemma. □

Following [Wu, Definition 3.15], a diagram

$$0 \longrightarrow W_0 \xrightarrow{i} V \xrightarrow{p} W_1 \longrightarrow 0$$

of diffeological vector spaces is a **short exact sequence of diffeological vector spaces** if it is a short exact sequence of vector spaces, i is an induction, and p is a subduction.

Lemma 3.26. *Let*

$$0 \longrightarrow W_0 \xrightarrow{i} V \xrightarrow{p} W_1 \longrightarrow 0$$

be a short exact sequence of diffeological vector spaces. If W_0 is indiscrete, then the sequence splits smoothly, so that $V \cong W_0 \oplus W_1$ as diffeological vector spaces.

Proof. Let $q : V \to W_0$ be any linear function such that $q \circ i = \mathrm{id}_{W_0}$. Since W_0 is indiscrete, q is smooth. Let $k : V \to V$ be the smooth linear map sending v to $v - i(q(v))$. Then $k \circ i = 0$, so k factors as $j \circ p$, where $j : W_1 \to V$ is smooth and linear. The smooth bijection $V \to W_0 \oplus W_1$ sending v to $(q(v), p(v))$ has a smooth inverse sending (w_0, w_1) to $i(w_0) + j(w_1)$, so the claim follows. □

It follows that many properties of a diffeological vector space are equivalent in this setting:

Proposition 3.27. *Let V be in $\mathcal{DV}$. Then the following are equivalent:*

(1) *V is in $\mathcal{SV}$.*
(2) *V is in $\mathcal{SD}$.*
(3) *V is in $\mathcal{FFV}$.*
(4) *$D(V)$ is Hausdorff.*
(5) *V has no non-zero indiscrete subspace.*

Moreover, being in $\mathcal{DV}$ and satisfying one of these conditions is equivalent to being a subspace of a product of copies of $\mathbb{R}$.

Proof. Without any assumption on V, we have (1) $\implies$ (2) $\implies$ (3) and (2) $\implies$ (4) using Theorem 3.20 and Proposition 3.14. It is easy to see that (3) $\implies$ (5) and (4) $\implies$ (5). By Proposition 3.25, (5) $\implies$ (1) when V is in $\mathcal{DV}$, and so we have shown that the five conditions are equivalent for $V \in \mathcal{DV}$.

For the last claim, a product of copies of $\mathbb{R}$ is in both $\mathcal{SV}$ and $\mathcal{DV}$, and both are closed under taking subspaces. Conversely, if V is in $\mathcal{SV} \cap \mathcal{DV}$, it is easy to check that

$$V \to \prod_{L^\infty(V,\mathbb{R})} \mathbb{R}$$

defined by $v \mapsto (f(v))_{f \in L^\infty(V,\mathbb{R})}$ is a linear induction, and hence V is a subspace of a product of copies of $\mathbb{R}$. □

Remark 3.28.

(1) It is not true that every diffeological vector space is in $\mathcal{DV}$. For example, when $\mathbb{R}$ is equipped with the continuous diffeology (see Example 3.16), all smooth linear functionals are zero, but the diffeology is not indiscrete.
(2) Other properties we have studied cannot be added to Proposition 3.27. For example, we saw in Remark 3.13(1) that $\prod_\omega \mathbb{R}$ is in $\mathcal{SV}$ but is not fine or projective. And it is easy to see that $\prod_\omega \mathbb{R}$ is in $\mathcal{DV}$.

It is not hard to show that every fine diffeological vector space is in $\mathcal{DV}$. As a final example, we will show that not every projective diffeological vector space is in $\mathcal{DV}$, and therefore that none of our other conditions on a diffeological vector space V implies that V is in $\mathcal{DV}$.

We will again use the diffeological vector space $F(\mathbb{R})$, which is projective by [Wu, Corollary 6.4]. We now show that it is not in $\mathcal{DV}$.

Proposition 3.29. *The free diffeological vector space $F(\mathbb{R})$ generated by $\mathbb{R}$ is not in $\mathcal{DV}$.*

Proof. Fix a non-zero smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp}(\phi) \subset (0,1)$ and $|\phi(x)| \leq 1$ for all $x \in \mathbb{R}$. For each $n \in \mathbb{Z}^+$, define $\phi_n : \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) = \phi\left(\frac{x - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right).$$

Finally, define $g : \mathbb{R} \to F(\mathbb{R})$ by

$$g(t) = \begin{cases} 2^{-n}\, \phi_n(t)\, \sum_{i=1}^n [\frac{1}{i}], & \text{if } \frac{1}{n+1} \leq t < \frac{1}{n}, \text{ for } n > 0 \\ 0, & \text{else.} \end{cases}$$

Then g is not a plot of $F(\mathbb{R})$, since locally around $0 \in \mathbb{R}$, g cannot be written as a finite sum of $f_i(x)[h_i(x)]$, where f_i and h_i are smooth functions with codomain $\mathbb{R}$. But for each $l \in L^\infty(F(\mathbb{R}),\mathbb{R})$,

$$l \circ g(t) = \begin{cases} 2^{-n}\, \phi_n(t)\, \sum_{i=1}^n l([\frac{1}{i}]), & \text{if } \frac{1}{n+1} \leq t < \frac{1}{n} \\ 0, & \text{else.} \end{cases}$$

This is smooth, since the set $\{l([\frac{1}{i}])\}$ is bounded, using the smoothness of l. □

As an easy corollary, we have:

Corollary 3.30. *$F(\mathbb{R})$ is not a subspace of a product of copies of $\mathbb{R}$.*

4. Some applications

Recall that a diagram

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$$

is a short exact sequence of diffeological vector spaces if it is a short exact sequence of vector spaces such that f is an induction and g is a subduction. We say that the sequence **splits smoothly** if there exists a smooth linear map $r : V_2 \to V_1$ such that $r \circ f = 1_{V_1}$, or equivalently, if there exists a smooth linear map $s : V_3 \to V_2$ such that $g \circ s = 1_{V_3}$. In either case, V_2 is smoothly isomorphic to $V_1 \times V_3$. (See [Wu, Theorem 3.16].)

Not every short exact sequence of diffeological vector spaces splits smoothly. For example, if we write K for the subspace of $C^\infty(\mathbb{R}, \mathbb{R})$ consisting of the smooth functions which are flat at 0, then K is not a smooth direct summand of $C^\infty(\mathbb{R}, \mathbb{R})$ [Wu, Example 4.3].

As a first application of the theory established so far, we can construct short exact sequences of diffeological vector spaces which do not split smoothly:

Example 4.1. Let M be a manifold of positive dimension, and let A be a finite subset of M. Write V for the subspace of $F(M)$ spanned by the subset $M \setminus A$ of M. We claim that V is not a smooth direct summand of $F(M)$.

To see this, write W for the quotient diffeological vector space $F(M)/V$. Then, as a vector space, $W = \oplus_{a \in A} \mathbb{R}$. So we have a short exact sequence $V \to F(M) \to W$ in DVect. Suppose this sequence splits smoothly. By Example 3.5, $F(M)$ is projective, and therefore W is as well. By Proposition 3.12 and Theorem 3.18, W is in $\mathcal{FFV}$. Since W is finite-dimensional, it is fine. But the smooth map $M \to F(M) \to W = \oplus_{a \in A} \mathbb{R}$ sends each $a \in A$ to a basis vector and other points in M to 0, so it is not a smooth map in the usual sense. This contradicts the fact that W is fine.

As a second application, we prove:

Theorem 4.2. *Let V be in $\mathcal{SV}$. Then every finite-dimensional subspace of V is a smooth direct summand.*

Proof. Let W be a finite-dimensional subspace of $V \in \mathcal{SV}$. By Theorem 3.18, we know that W has the fine diffeology. Moreover, since V is in $\mathcal{SV}$, there is a smooth linear injective map $V \to \prod_{i \in I} \mathbb{R}$ for some index set I. Since $\prod_{i \in I} \mathbb{R}$ is in $\mathcal{SV}$, again by Theorem 3.18, we know that the composite $W \hookrightarrow V \to \prod_{i \in I} \mathbb{R}$ is an induction, although the second map might not be an induction. So, we are left to prove this statement for the case $V = \prod_{i \in I} \mathbb{R}$.

Write $\dim(W) = m$. By Gaussian elimination, there exist distinct $i_1, \ldots, i_m \in I$ such that the composite $W \hookrightarrow V = \prod_{i \in I} \mathbb{R} \to \mathbb{R}^m$ is an isomorphism of vector spaces, where the second map is the projection onto the $i_1, \ldots, i_m$ coordinates, and hence smooth. Since both W and $\mathbb{R}^m$ have the fine diffeology, this isomorphism is a diffeomorphism, and by composing with its inverse we obtain a smooth linear map $r : V \to W$ such that the composite

$$W \hookrightarrow V \xrightarrow{r} W$$

is 1_W. Therefore, W is a smooth direct summand of V. □

5. Proof of Theorem 3.20

Theorem 3.20. *Every diffeological vector space in $\mathcal{SD}$ is in $\mathcal{FFV}$.*

Proof. If a diffeological vector space is in $\mathcal{SD}$, then so are all of its subspaces. So it suffices to show that every finite-dimensional diffeological vector space in $\mathcal{SD}$ is fine.

Write V for $\mathbb{R}^n$ with the structure of a diffeological vector space which is not fine. We will use the word "smooth" (resp. "continuous") to describe functions $\mathbb{R} \to V$ and $V \to \mathbb{R}$ which are smooth (resp. continuous) with respect to the usual diffeology (resp. topology) on $\mathbb{R}^n$. We use the word "plot" to describe functions $\mathbb{R} \to V$ which are in the diffeology on V, and write $f \in C^\infty(V, \mathbb{R})$ to describe functions which are smooth with respect to this diffeology.

By Proposition 3.3, there is a plot $p : \mathbb{R} \to V$ which is not smooth. Since plots are closed under translation in the domain and codomain, we can assume without loss of generality that $p(0) = 0$ and p is not smooth at $0 \in \mathbb{R}$. We will show that this implies that V is not in $\mathcal{SD}$.

Case 1: Suppose that p is continuous at 0. Consider $A := \{\nabla f(x) \mid f \in C^\infty(V, \mathbb{R}), x \in V\}$. Then A is a subset of $\mathbb{R}^n$.

We claim that A is a proper subset of $\mathbb{R}^n$. If A is not proper, then there exist $(f_1, a_1), \ldots, (f_n, a_n) \in C^\infty(V, \mathbb{R}) \times V$ such that $\nabla f_1(a_1), \ldots, \nabla f_n(a_n)$ are linearly independent. Then $g_i : V \to \mathbb{R}$ defined by $g_i(x) = f_i(x + a_i)$ is in $C^\infty(V, \mathbb{R})$, $G := (g_1, \ldots, g_n) : V \to \mathbb{R}^n$ is smooth, and the Jacobian $JG(0)$ is invertible. Therefore, by the inverse function theorem, G is a local diffeomorphism near $0 \in V$. Since $p(0) = 0$, p is continuous at $0 \in \mathbb{R}$, and $G \circ p$ is smooth, it follows that p is smooth at 0, contradicting our assumption on p. So A is a proper subset.

By the same method of translation, one sees that A is a subspace of $\mathbb{R}^n$. Hence, there exists $v \in \mathbb{R}^n$ such that $v \perp A$, which implies that $f(x + tv) = f(x)$ for every $f \in C^\infty(V, \mathbb{R})$, $x \in V$ and $t \in \mathbb{R}$, i.e., V is not in $\mathcal{SD}$.

Case 2: Suppose that p is not continuous at 0.

Case 2a: Suppose there exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that $t^k p(t)$ is bounded on $[-\epsilon, \epsilon]$. Let k be the smallest such exponent and write $q(t) := t^k p(t)$, which is also a plot. We claim that q is not smooth at 0. If $k = 0$, then $q = p$, which is assumed to not be smooth at 0. If $k > 0$ and $q'(0)$ exists, then $q(t)/t \to q'(0)$ as $t \to 0$, which implies that $t^{k-1}p(t)$ is also bounded on $[-\epsilon, \epsilon]$, contradicting the minimality of k. So q is not smooth at 0.

If q is continuous at 0, then by Case 1, we are done.

So assume that q is not continuous at 0. Then, since q is bounded on $[-\epsilon, \epsilon]$, there exists a sequence t_i converging to 0 such that $q(t_i)$ converges to a non-zero $v \in V$. If f is in $C^\infty(V, \mathbb{R})$, then $f \circ q$ is smooth, so $f(0) = f(q(0)) = f(q(\lim t_i)) = \lim f(q(t_i)) = f(\lim q(t_i)) = f(v)$. Therefore, the functions in $C^\infty(V, \mathbb{R})$ do not separate points.

Case 2b: Suppose that Case 2a does not apply. Then for each $k \in \mathbb{N}$, $\epsilon > 0$ and $M > 0$, there exists $t \in [-\epsilon, \epsilon]$ such that $\|t^k\, p(t)\| > M$. (Note that $t \neq 0$, since $p(0) = 0$.) Using this for $k = 0$, choose $t_1 \in [-1, 1]$ such that $\|p(t_1)\| > 1$. Then, for each integer $k > 0$, choose t_k with $|t_k| \leq |t_{k-1}|/2$ such that $\|t_k^k\, p(t_k)\| > k$. If $m \leq k$, then t_k also satisfies $\|t_k^m\, p(t_k)\| > k \geq m$, since $|t_k| \leq 1$. Therefore, we can restrict to a subsequence of the t_k all having the same sign. To fix notation, assume that each t_k is positive. Then, for $m \leq k$,

$$\frac{\frac{1}{\|p(t_k)\|}}{t_k^m} < \frac{1}{k}$$

and so, for each m, the left-hand-side goes to 0 as $k \to \infty$. By Lemma 5.1 below, there is a smooth curve $c : \mathbb{R} \to \mathbb{R}$ such that $c(t_k) = 1/\|p(t_k)\|$. It follows that $q(t) := c(t)p(t)$ is a plot such that $q(0) = 0$ but $q(t_k)$ is bounded as $k \to \infty$ (since $\|q(t_k)\| = 1$). Therefore, there is a subsequence converging to a non-zero $v \in V$, and the argument at the end of Case 2a shows that $C^\infty(V, \mathbb{R})$ does not separate points. $\square$

Lemma 5.1 (Extended special curve lemma). *Let $\{x_k\}$ and $\{t_k\}$ be sequences in $\mathbb{R}$ such that $0 < t_k < t_{k-1}/2$ for each k and $x_k/t_k^m \to 0$ as $k \to \infty$ for each $m \in \mathbb{Z}^+$. Then there is a smooth function $c : \mathbb{R} \to \mathbb{R}$ such that $c(t_k) = x_k$ for each k and $c(t) = 0$ for $t < 0$.*

The proof closely follows [KM, page 18], and can easily be generalized further.

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$. Define $c : \mathbb{R} \to \mathbb{R}$ by

$$c(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ x_{k+1} + \phi\left(\frac{t-t_{k+1}}{t_k - t_{k+1}}\right)(x_k - x_{k+1}), & \text{for } t_{k+1} \leq t \leq t_k \\ x_1, & \text{for } t_1 \leq t. \end{cases}$$

c is smooth away from 0 and for $t_{k+1} \leq t \leq t_k$ we have

$$c^{(r)}(t) = \phi^{(r)}\left(\frac{t - t_{k+1}}{t_k - t_{k+1}}\right)\frac{1}{(t_k - t_{k+1})^r}(x_k - x_{k+1}).$$

Since $t_k - t_{k+1} > t_k/2 > t_{k+1}$, the right-hand-side goes to zero as $t \to 0$. Similarly, $c^{(r)}(t)/t \to 0$, which shows that each $c^{(r+1)}(0)$ exists and is 0. So c is smooth. □

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