

A DUAL NON-ABELIAN YANG-MILLS AMPLITUDE IN FOUR DIMENSIONS

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ABSTRACT. We derive an explicit formula for the vertex amplitude of dual $SU(2)$ Yang-Mills theory in four dimensions on the lattice, and provide an efficient algorithm (of order j^4) for its computation. This opens the way for both numerical and analytical development of dual methods, previously limited to the case of three dimensions.

1. INTRODUCTION

Since their inception in the early 1970's, quantum field theories formulated on the lattice have been mapped to dual theories — models mathematically equivalent to the original formulation, but with different degrees of freedom. In many cases, configurations in the dual model can be interpreted as extended geometric structures such as closed sheets or self-avoiding polymers living on the space-time lattice [1, 2, 3]; such structures can in turn be interpreted as histories of the physical degrees of freedom of the theory [4]. Recently, the concept of dual methods as an alternative framework for computations in lattice gauge theory has seen a renewal of interest [5, 6]. Apart from the geometric insight provided by the dual theories and their use of gauge-invariant physical degrees of freedom, dual methods may provide a source of novel algorithms with attractive numerical performance in some regimes. From early work (e.g. [7]) it was found that abelian gauge theories are particularly tractable in their dual form. The abelian case was initially a source of many dual results and continues to provide a practical alternative to conventional numerical simulations [8, 9, 10, 11, 12]. With the above motivations in mind, we turn to the dual model of $SU(2)$ Yang-Mills theory on the four-dimensional lattice. This case is an important proving ground for the viability of dual non-abelian method, and one that has long been out of reach for practical calculations.

In analogy with dual abelian constructions, a dual formulation for pure non-abelian Yang-Mills lattice gauge theory (LGT) in four dimensions can be constructed from the character expansion of a local lattice action and has been known for decades (e.g. see [13] and references therein). However, the naïve application of character expansion leads to a non-local expression for the amplitude — a group integral over the entire lattice. It has since been realized by several authors that this group integral, which involves products of characters, can be expanded into intertwiner labellings at lattice edges, such that the dual amplitude becomes a product of amplitudes *local* in the plaquette and intertwiner labellings. Following the description of an explicitly local amplitude in the case of $D = 3$ and $G = SU(2)$ by Anishetty *et al.* in [14, 15, 16], Halliday and Suranyi observed in [17] that a local amplitude in four and higher dimensions can in principle be defined. However, a practical formula for the dual amplitude in $D = 4$ (analogous to that of [14, 15, 16]) has proven more elusive.

Recently, a systematic approach to non-abelian dual theories that emphasizes the underlying representation-theoretic structure has been developed starting with work by Oeckl and Pfeiffer [18, 19]. These developments make use of the diagrammatic formalism to establish explicit representation-theoretic formulae for the local amplitudes that arise upon expanding in an intertwiner basis.

It was demonstrated in [20] that numerical computations could be carried out for the dual form of LGT with $G = SU(2)$ on a cubic three-dimensional ($D = 3$) lattice. One major hurdle to extending the algorithm of [20] to $D = 4$ is the problem of finding a closed-form formula for the vertex amplitude that can be evaluated efficiently on a computer and gives reasonable scaling with increasing spin labels.

In what follows, we present such an explicit and efficient evaluation of the vertex amplitude, a non-trivial function of 48 spin labels that dominates the computational costs of evaluating the dual amplitude in four dimensions. Our implementation of this formula has been validated for a range of special cases; currently, full Monte Carlo simulations using this amplitude are under way and will be described in an upcoming paper.

2. EXPLICIT ALGORITHM FOR THE VERTEX AMPLITUDE

2.1. Review. The steps we shall use to find an explicit formula for the vertex amplitude are analogous to those described in [20] for the case of $D = 3$. For a hypercubic lattice κ in four dimensions, we write the integral of the six $SU(2)$ matrices sharing a given edge variable g_e as follows:

$$(1) \quad \int dg_e U_{j_1}(g_e)_{a_1}^{b_1} U_{j_2}(g_e)_{a_2}^{b_2} U_{j_3}(g_e)_{a_3}^{b_3} U_{j_4}(g_e^{-1})_{a_4}^{b_4} U_{j_5}(g_e^{-1})_{a_5}^{b_5} U_{j_6}(g_e^{-1})_{a_6}^{b_6} = \int dg_e \begin{array}{c} \leftarrow \textcircled{g_e} \leftarrow j_1 \\ \leftarrow \textcircled{g_e} \leftarrow j_2 \\ \leftarrow \textcircled{g_e} \leftarrow j_3 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_4 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_5 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_6 \end{array},$$

where j_1, \dots, j_6 denote irreducible representations of $SU(2)$, which we index by non-negative half-integers $0, \frac{1}{2}, 1, \dots$. The RHS of (1) is a projection map from the space of all linear maps $j_6 \otimes j_5 \otimes j_4 \rightarrow j_1 \otimes j_2 \otimes j_3$ onto the space of intertwiners (maps commuting with the action of G), and can thus be expanded using an orthogonal basis of intertwiners $I_i : j_6 \otimes j_5 \otimes j_4 \rightarrow j_1 \otimes j_2 \otimes j_3$ as follows:

$$(2) \quad \int dg_e \begin{array}{c} \leftarrow \textcircled{g_e} \leftarrow j_1 \\ \leftarrow \textcircled{g_e} \leftarrow j_2 \\ \leftarrow \textcircled{g_e} \leftarrow j_3 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_4 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_5 \\ \rightarrow \textcircled{g_e^{-1}} \rightarrow j_6 \end{array} = \sum_i \frac{I_i I_i^*}{\langle I_i^*, I_i \rangle} = \sum_i \frac{\begin{array}{c} j_1 \leftarrow \\ j_2 \leftarrow \\ j_3 \leftarrow \\ j_4 \rightarrow \\ j_5 \rightarrow \\ j_6 \rightarrow \\ \textcircled{I_i} \textcircled{I_i^*} \end{array}}{\begin{array}{c} j_1 \leftarrow \\ j_2 \leftarrow \\ j_3 \leftarrow \\ j_4 \rightarrow \\ j_5 \rightarrow \\ j_6 \rightarrow \\ \textcircled{I_i^*} \textcircled{I_i} \end{array}}.$$

Carrying out the contractions results in a spin foam model (see, for example, [19, 20, 21] for details), with the vertex amplitude defined as a spin network evaluation of the network shown in Figure 1. In drawing

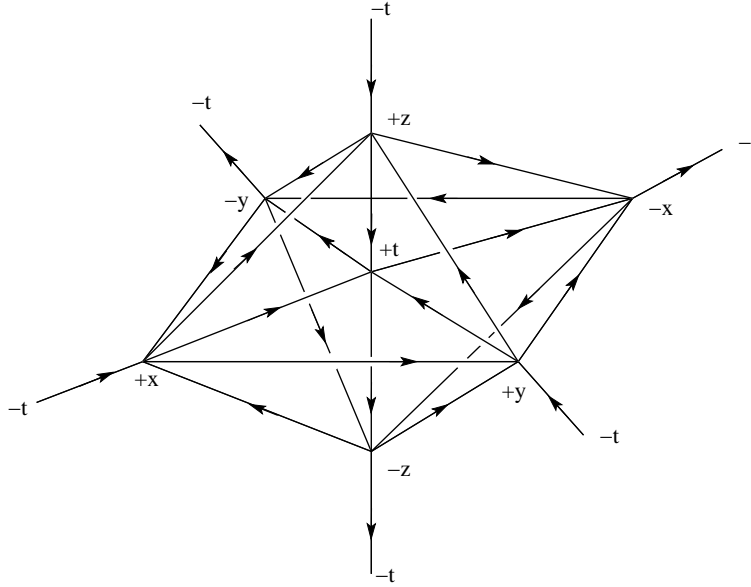


FIGURE 1. Vertex amplitude for hypercubic lattice in four dimensions.

this figure, we have placed the vertex labelled $-t$ “at infinity”. The edges of this network carry the irrep

labels j_i from the associated plaquettes meeting at the vertex and the vertices carry the intertwiner labels appearing in the expansion (2), associated to the edges incident to the vertex. The orientations of the edges are determined from an orientation of the plaquettes in the lattice. We note in passing that this network has 24 edges and 8 vertices, and shares the same graph structure as a “hyperoctahedron”.

To apply diagrammatic techniques leading to an explicit formula, our first task is to work with a diagram possessing 3-valent vertices, for which recoupling rules from [22, 23] can be applied. A basis for the 6-valent intertwiners appearing in (2) can be readily chosen in terms of 3-valent networks having six edges entering or leaving (see Figure 2). Because three internal edges are required, the edges of the four-dimensional lattice will carry three half-integer spin labels, subject to some constraints. (In contrast, for the 4-valent vertices arising in $D = 3$, only a single label is required to define the intertwiner assigned to an edge.)

Returning to the spin network that defines the vertex amplitude, we see that because each network vertex is associated with three spins, the vertex amplitude becomes a function of 48 spins in total; 24 from the network edges (coming from lattice plaquettes) and 24 from the 8 network vertices (3 from each of the 8 lattice edges). Thus, we shall refer to a vertex amplitude (with its vertex splitting specified) as a $48j$ symbol. It should be noted that different patterns of splittings over the edges of the lattice can be chosen leading to numerically different vertex amplitudes¹.

An important technical consideration is that the network resulting from an arbitrary splitting does not necessarily have translation symmetry over all the lattice vertices, as the splitting along one direction may not be related to the splitting in its opposite direction by a translation of the network. Indeed, this is the case for the choice of splitting we will introduce below. However, a pattern of partial reflections leading to $2^4 = 16$ “versions” of the $48j$ symbol can be repeated to fill a lattice². As the sixteen versions are related to each other by appropriate permutations of arguments, it is sufficient to work out a single case, which we shall by convention define to be the $48j$ symbol at the origin³.

In what follows we present the explicit evaluation of the $48j$ symbol associated with a particular splitting (choice of intertwiner basis) that we give in the next section.

2.2. Step 1 — A splitting. We now specify the splitting we shall use for our 6-valent intertwiners. First we recall that a 3-valent intertwiner is uniquely determined up to a scale factor, and to fix conventions we use the intertwiners defined in [22]. Figure 2 shows a 6-valent intertwiner I and its dual I^* . As the internal

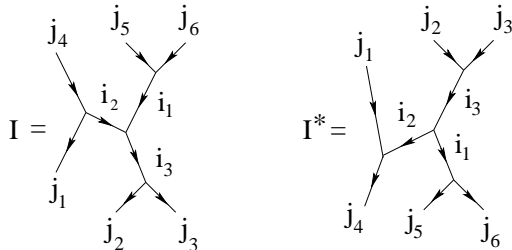


FIGURE 2. Our choice of 6-valent intertwiners: a “three-fold” splitting.

labels i_1 , i_2 and i_3 vary, the intertwiners I form an orthogonal basis of all 6-valent intertwiners, and these are the splittings used in our recoupling of the vertex amplitude below. Observe here that the six incident edges are grouped into three pairs; the choices we make are shown in Figure 3, in which each 6-valent vertex has been replaced by an intertwiner I as in Figure 2. The central vertex of I is labelled with the same label as the original 6-valent vertex had.

Before we can apply recoupling rules to explicitly compute the vertex amplitude spin network, we must cast it into the form where the rules of [22, 23] can be applied. To do this we follow the procedure described in Appendix A of [20], which relates the original spin network (with oriented edges) defining the $48j$ symbol

¹Of course, for a given labelling of plaquettes, the sum over intertwiner labels of the product of edge and vertex amplitudes over the lattice is independent of the choice of splitting.

²The lattice must have even side-lengths, which is not a serious constraint in practice.

³A similar phenomenon can arise in the three-dimensional case [24], although there it is sufficient to use two closely related network evaluations; alternatively, a translation invariant splitting can be used [20].

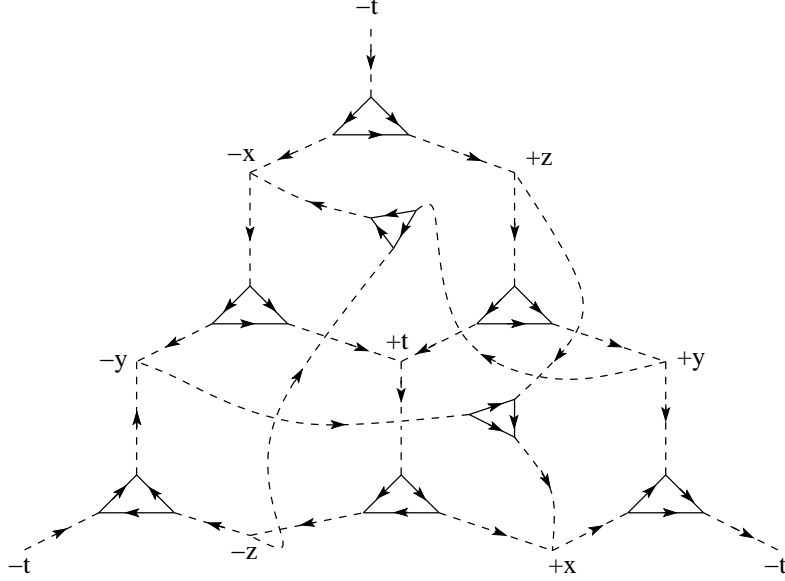


FIGURE 3. A spin network diagram representing the $48j$ symbol in our choice of splitting.

to a spin network in the sense of [22, 23]; although the two are equal in magnitude, in general there is a sign factor that depends on the spin arguments.

To determine the sign factor, we start by redrawing the network with all edges directed down the page, except for some edges which wrap around the right to the top of the page and contribute a sign factor. In doing this, the order of the edges incident on each vertex must be kept the same, but over-crossings and under-crossings can be reversed. Such a mapping is presented in Figure 4. We first recall each line wrapping

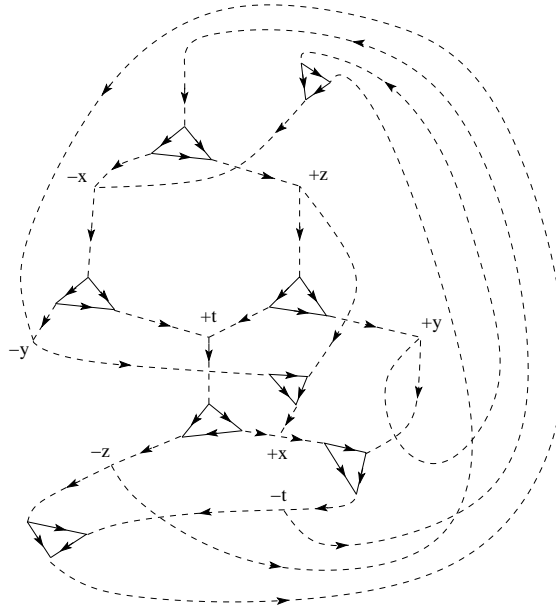


FIGURE 4. Redrawing of vertex amplitude to determine sign factor.

around contributes a sign factor of $(-1)^{2i}$. We next observe that each intertwiner label that appears in the overall sign factor $(-1)^{2\sum i}$ for a given vertex will also appear in the sign factor of the vertex on the other side of the edge. Thus, globally these sign factors will cancel. Because of this, we are free to “drop the orientation” of the original spin network and apply the rules of [22] and [23], as was done in [20].

2.3. Step 2 — Collapse of triangles. Referring to the splitting we have indicated in Figure 3, we have drawn the diagram so as to make clear eight triangles, the edges of which are labelled by spins coming from plaquettes, and the vertices of which connect to edges carrying intertwiner labels. The next step in our recoupling procedure is to collapse each of these by applying the identity

$$(3) \quad \begin{array}{c} e \\ \vdots \\ b \quad c \\ \triangle \\ f \\ \vdots \\ a \quad d \end{array} = \frac{\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}}{\theta(a, d, e)}, \quad \begin{array}{c} e \\ \vdots \\ \quad \quad \quad \\ \vdots \\ a \quad d \end{array},$$

which introduces the Kauffman-Lins tet symbol (the square bracketed expression) and theta network ⁴.

After collapsing the triangles, the vertex amplitude A_v is given by

$$(4) \quad A_v[j_1, j_2, \dots, j_{24}, i_1, i_2, \dots, i_{24}] = \tau_1 \tau_2 \left(\prod_{n=1}^8 \frac{\begin{bmatrix} i_{a_1(n)} & j_{b_1(n)} & i_{e_1(n)} \\ j_{c_1(n)} & i_{d_1(n)} & j_{f_1(n)} \end{bmatrix}}{\theta(i_{a_1(n)}, i_{d_1(n)}, i_{e_1(n)})} \right) S_0,$$

where S_0 is the spin network depicted in Figure 5. In Figure 5 it should be noted that, relative to Figure 4, two twists (indicated by bars spanning twisted strands) have been introduced, giving rise to associated sign factors τ_1 and τ_2 (see Step 3 below for discussion). This was to put the diagram in a particular planar order so that the identity (5) defined in the next section below can be applied unambiguously.

An interesting remark at this point is that the dependence on the 24 plaquette spins j_i has been completely absorbed into the product of collapse factors in (4); the remaining spin network is purely a function of the 24 intertwiner labels.

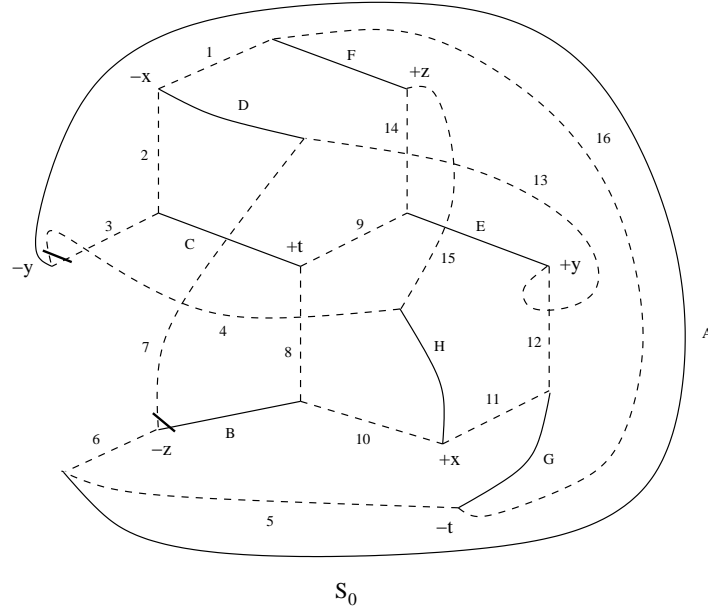


FIGURE 5. Spin network evaluation after removing triangles. In Step 3, recoupling moves are applied to solid lines.

⁴As discussed in [20], the tet network is closely related to the Racah-Wigner $6j$ symbol. Note that [23] uses integer spins where we and [22] use half-integer spins.

2.4. Step 3 — Transforming to a ladder. In this step we transform the spin network S_0 into a ladder via a sequence of eight recoupling moves of the form

$$(5) \quad \begin{array}{c} b \\ \diagdown \\ \text{---} R \text{---} \\ \diagup \\ a \end{array} \begin{array}{c} c \\ \diagup \\ \text{---} \\ \diagdown \\ d \end{array} = \frac{\Delta_m \begin{bmatrix} a & b & m \\ c & d & R \end{bmatrix}}{\theta(a, d, m)\theta(b, c, m)} \begin{array}{c} b \\ \diagdown \\ \text{---} \\ \diagup \\ a \end{array} \begin{array}{c} c \\ \diagup \\ \text{---} m \text{---} \\ \diagdown \\ d \end{array},$$

where $\Delta_m = (-1)^{2m}(2m+1)$. The moves are indicated in Figure 5 as follows. The eight edges to be recoupled are drawn as solid lines and labelled with letters from A through H . As given by (5), each of these moves leads to an auxiliary sum over an additional irrep label m_i . Upon applying the recoupling moves one arrives at the spin network shown in Figure 6(a).

While these moves are sufficient to give a network having the topology of a ladder, there are a number of vertices at which “twist moves” have to be performed to arrive at a ladder without crossings. These twists do not effect the magnitude of the spin network evaluation but can introduce signs, as follows:

$$(6) \quad \begin{array}{c} \alpha \\ \diagdown \\ \text{---} \\ \diagup \\ \gamma \end{array} \begin{array}{c} \beta \\ \diagup \\ \text{---} \\ \diagdown \\ \gamma \end{array} = \tau(\alpha, \beta, \gamma) \begin{array}{c} \alpha \\ \diagdown \\ \text{---} \\ \diagup \\ \gamma \end{array} \begin{array}{c} \beta \\ \diagup \\ \text{---} \\ \diagdown \\ \gamma \end{array},$$

where $\tau(\alpha, \beta, \gamma) = (-1)^{\alpha+\beta-\gamma}$. The resulting expression for S_0 is

$$(7) \quad S_0[i_1, i_2, \dots, i_{24}] = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_8} \left(\prod_{n=1}^8 \frac{\Delta_{m_n} \tau_n \begin{bmatrix} i_{a_2(n)} & i_{b_2(n)} & m_n \\ i_{c_2(n)} & i_{d_2(n)} & i_{R(n)} \end{bmatrix}}{\theta(i_{a_2(n)}, i_{d_2(n)}, m_n)\theta(i_{b_2(n)}, i_{c_2(n)}, m_n)} \right) S_1,$$

where $\tau_n = \tau(i_{\alpha(n)}, i_{\beta(n)}, m_n)$ are the factors associated with the twists that are attached to the eight recoupled edges. S_1 is the ladder spin network shown in Figure 6(b). The spins m_1 through m_8 are associated to the new edges, labelled A' through H' . The eight intertwiners that labelled the edges A through H in Figure 5, denoted R in the recoupling move (5), appear as $i_{R(n)}$ in (7).

2.5. Step 4 — Collapsing the ladder. Here we apply the recoupling move to the eight “rungs” of the ladder. Superficially, this leads to an additional eight auxiliary summations. However, upon removing the bubbles (which are proportional to the identity intertwiner) it becomes clear that all eight auxiliary spins must be equal for the resulting network to be non-zero (Schur’s Lemma). Thus, only a *single* additional auxiliary summation is required. Hence

$$(8) \quad S_1[i_1, i_2, \dots, i_{16}, m_1, \dots, m_8] = \sum_l \left(\prod_{n=1}^8 \frac{\Delta_l \begin{bmatrix} i_{a_3(n)} & m_{b_3(n)} & l \\ i_{c_3(n)} & m_{d_3(n)} & i_{r(n)} \end{bmatrix}}{\theta(i_{a_3(n)}, m_{d_3(n)}, l)\theta(i_{b_3(n)}, m_{c_3(n)}, l)} \right) S_2,$$

where S_2 is the chain of bubbles; note here that the $i_{r(n)}$ are the intertwiners corresponding to the “rungs” of the ladder in Figure 6(b).

To complete the evaluation we apply the “bubble collapse” move

$$(9) \quad \begin{array}{c} l \\ | \\ a \text{---} \text{---} b \\ | \\ l \end{array} = \frac{\theta(a, b, l)}{\Delta_l} \left| \begin{array}{c} l \\ | \\ l \end{array} \right.$$

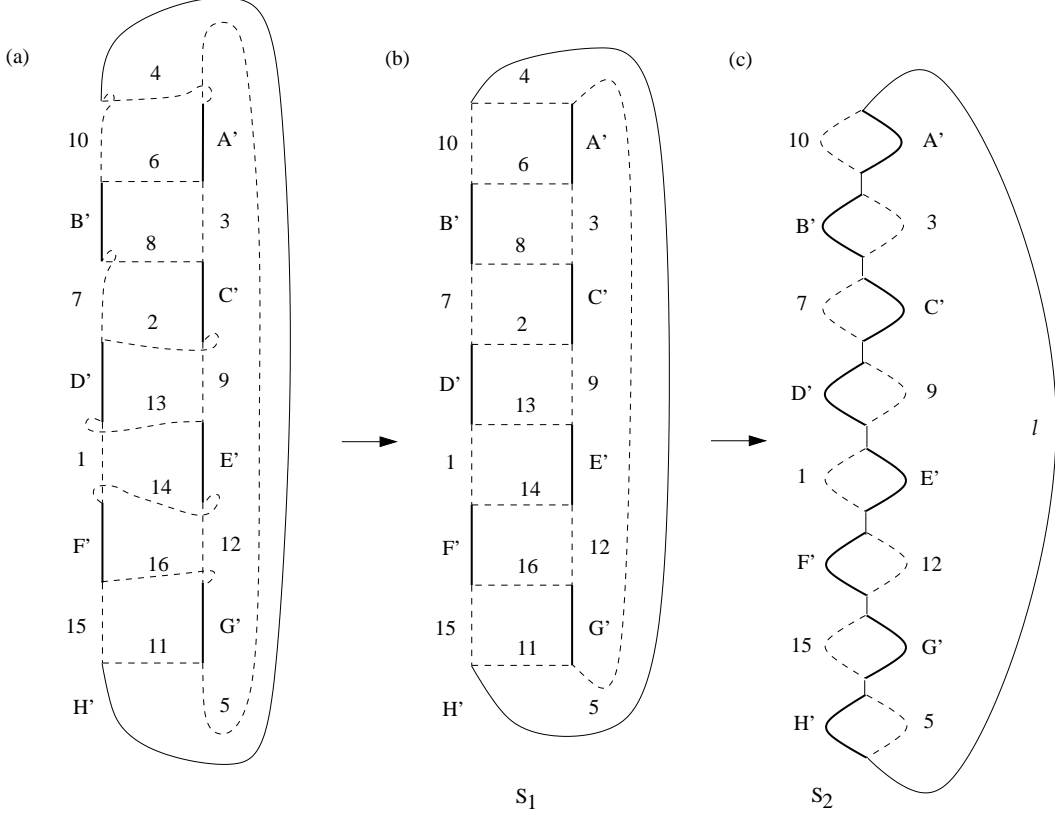


FIGURE 6. Final steps in recoupling, leading to a matrix multiplication of tet networks.

to each of the eight bubbles in Figure 6(c), which results in a product of theta and delta factors:

$$(10) \quad S_2[i_1, i_2, \dots, i_8, m_1, \dots, m_8] = \Delta_l \left(\prod_{n=1}^8 \frac{\theta(i_n, m_n, l)}{\Delta_l} \right).$$

The Δ_l factor in front of the product comes from the loop formed after transforming the eight bubbles.

Assembling the results of our previous steps yields the following:

$$(11) \quad A_v[j_1, j_2, \dots, j_{24}, i_1, i_2, \dots, i_{24}] = \left(\prod_{n=1}^8 \begin{bmatrix} i_{a_1(n)} & j_{b_1(n)} & i_{e_1(n)} \\ j_{c_1(n)} & i_{d_1(n)} & j_{f_1(n)} \end{bmatrix} \right) \sum_l \sum_{m_1} \sum_{m_2} \dots \sum_{m_8} \Delta_l \left(\prod_{n=1}^8 \frac{\Delta_{m_n} \tau_n \begin{bmatrix} i_{a_2(n)} & i_{b_2(n)} & m_n \\ i_{c_2(n)} & i_{d_2(n)} & i_{R(n)} \end{bmatrix} \begin{bmatrix} i_{a_3(n)} & m_{b_3(n)} & l \\ i_{c_3(n)} & m_{d_3(n)} & i_{r(n)} \end{bmatrix}}{\theta(i_{a_2(n)}, i_{d_2(n)}, m_n) \theta(i_{b_2(n)}, i_{c_2(n)}, m_n) \theta(i_{a_3(n)}, m_{d_3(n)}, l) \theta(i_{b_3(n)}, m_{c_3(n)}, l)} \theta(i_n, m_n, l) \right),$$

where the Δ_l factors appearing in the denominator of (10) cancel with the Δ_l factors in (8).

2.6. Computational Cost. Suppose we choose a spin cut-off of j , so all spin and intertwiner labels must be at most j . We will estimate the time required to compute the vertex amplitude (11).

The factors in (11) each depend on only one m_n variable, except for the third tet network, which depends on two of the m_n variables. In fact, one can show that this dependence is in a cyclical order, so that, for fixed l , the vertex amplitude can be computed as the trace of a product of eight matrices. The range of the m_n variables is restricted by the triangle equalities to be $O(j)$, so the matrices have size $O(j) \times O(j)$. The entries of these matrices are products of tet networks and θ networks, which require $O(j)$ time to compute. Thus forming the matrices and multiplying them together each require $O(j^3)$ time. Considering the sum over l , which is also constrained to be $O(j)$, we conclude that the overall cost to computing the vertex amplitude is $O(j^4)$.

For comparison with the $D = 3$ case, we recall that the amplitude introduced by [14, 15, 16] and used in [24, 25, 26] was $O(j)$. This $O(j)$ algorithm as well as an alternative $O(j^2)$ vertex amplitude were used in the $D = 3$ dual LGT simulations [20].

2.7. Edge amplitude. To complete the model, we describe how the normalization factor appearing in the denominator on the RHS of (2) can be given an explicit expression. Having chosen the three-fold splitting described above for our intertwiners, we draw the edge amplitude evaluation as shown in Figure 7(a). As

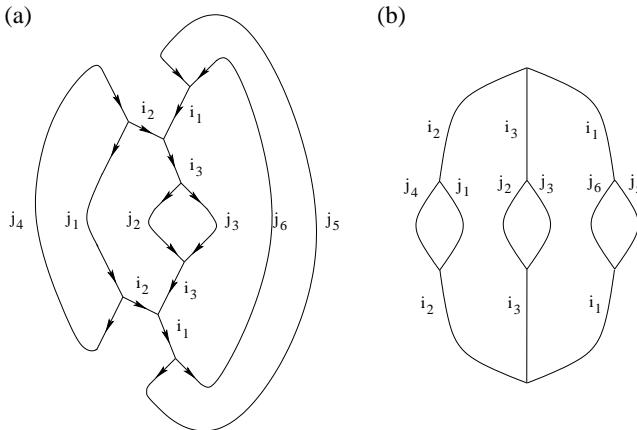


FIGURE 7. Normalization of edges.

above, one can compute the sign relating the original network with directed edges to an undirected Kauffman-Lins spin network, and once again one finds that globally these signs cancel. The undirected spin network is shown in Figure 7(b) and can be transformed into a theta network by collapsing the three bubbles according to the identity (9), as follows:

$$(12) \quad A_e[j_1, j_2, j_3, j_4, j_5, j_6, i_1, i_2, i_3] = \frac{\theta(j_4, j_1, i_2)\theta(j_2, j_3, i_3)\theta(j_6, j_5, i_1)}{\Delta_{i_1}\Delta_{i_2}\Delta_{i_3}}\theta(i_1, i_2, i_3).$$

Here j_1, \dots, j_6 are the labels for the six plaquettes incident to an edge and i_1, i_2, i_3 are the three intertwiner labels. Figure 7 also makes it clear that different intertwiners are orthogonal.

2.8. Validation of the Vertex and Edge Amplitudes. The best comprehensive check for the validity of the amplitude derived above is a full simulation of the dual theory at increasingly large cut-offs. While this work is currently in progress, a more limited form of validation has been done. In the case where dual configurations are ensembles of closed, self-avoiding surfaces on the lattice, there is an explicit closed-form expression for the resulting amplitude related to the coloring of self-avoiding surfaces by irreps and their Euler characteristic; this formula is derived in [21]. To verify that the amplitude was correct for these configurations, code was implemented to generate a range of such self-avoiding topologies (corresponding topologically to 2-tori and 2-spheres) and colorings. Using an implementation of the vertex and edge amplitudes given above, the amplitudes of self-avoiding dual configurations were found to be in agreement with the closed form formula.

3. CONCLUSIONS

We have presented an explicit formulation of pure Yang-Mills theory on a $D = 4$ hypercubic lattice with $G = SU(2)$. Computationally, the critical part of the amplitude is the vertex amplitude, a function of 48 spins coming from the 24 plaquette labels and 24 intertwiner labels associated to a vertex. In a variety of special cases where a closed form formula was available, the amplitude was successfully verified. The first results of dual simulations using this amplitude will be reported on in forthcoming work by the first author.

While the vertex amplitude scales with j^4 rather than j (as in the $D = 3$ case), the achievable performance is sufficient to enable the first practical four-dimensional computations within the dual model. This is an important step in evaluating the feasibility of dual algorithms for non-abelian LGT as a possible alternative to conventional methods. We would also like to remark that the discovery of the original $D = 3$ amplitude [14,

15, 16] prompted a number of intriguing analytic results. These include work by Diakonov and Petrov [27] revealing a gravitational interpretation to the dual degrees of freedom, as well as recent work by Conrady [28]. With the present result, the way is now clear to establish analogous analytic results in $D = 4$. In particular, it will be interesting to see which results of the above work are specific to $D = 3$ and which ideas can be extended $D = 4$.

4. ACKNOWLEDGEMENTS

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