

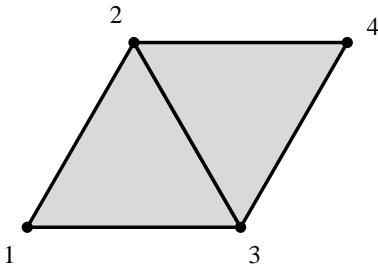
# DAVID GALE'S SUBSET TAKE-AWAY GAME

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**Subset take-away** is a two-player game involving a fixed finite set  $A$ . Players alternately choose proper, non-empty subsets of  $A$ , with the condition that one may not name a set containing a set that was named earlier. A player who is unable to move loses. For example, if  $A = \{1\}$ , then there are no legal moves and the second player wins. If  $A = \{1, 2\}$ , then the only legal moves are  $\{1\}$  and  $\{2\}$ . Each is a good reply to the other, and so once again the second player wins. The first interesting case is when  $A = \{1, 2, 3\}$ . In response to any first move, the second player may choose the complementary set. This produces a position equivalent to the starting position when  $A = \{1, 2\}$  and thus leads to a win for the second player. With increasing patience, the reader may enjoy verifying that when  $A$  has fewer than 6 elements, the game is a second player win. Indeed, David Gale, whom we understand deserves credit for this game, made the following conjecture [1].

**Conjecture 1.** *Subset take-away is always a second player win.*

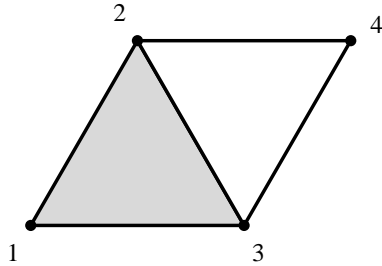
It was pointed out to us by Richard Stanley that the collection of legal moves at any given time forms an abstract simplicial complex: if  $X$  is a subset of  $A$  that is legal, then every non-empty subset of  $X$  is also legal. To translate this into geometry, view a  $k$ -subset as a  $(k - 1)$ -simplex. If  $A = \{1, 2, \dots, n\}$ , then the starting position corresponds to the boundary of the  $(n - 1)$ -simplex. A move in this formulation of the game consists of choosing a simplex of any dimension, and erasing its interior as well as all higher-dimensional simplices having it as a face. For example, the starting position for the  $n = 4$  game is a hollow tetrahedron, and after the move  $\{1, 4\}$  (which we write 14 for brevity) we have the position



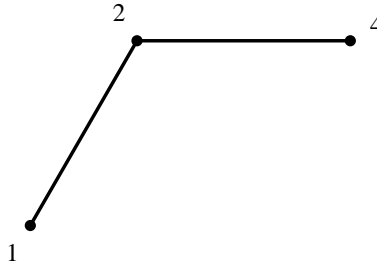
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This version is more up to date than the published version. See the last page for details.

in which the legal moves are 123 and 234 along with their non-empty subsets. Now one could remove the 2-simplex 234 leaving



And if the next player removed the vertex 3, the new position would be



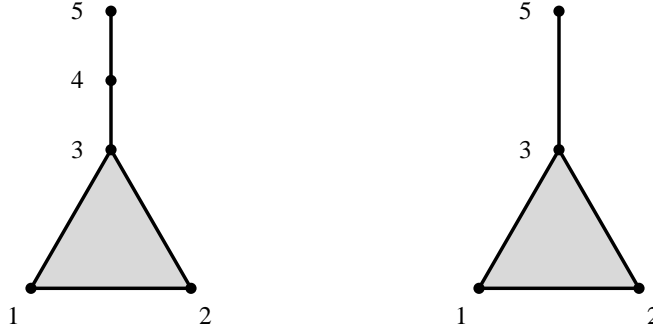
with five legal moves. A winning move here is to erase the vertex 2, leaving a game equivalent to the starting position for  $n = 2$ . The geometric formulation is quite helpful, and we will use this language frequently.

We next describe a technique that allows one to reduce the size of a position without changing its win/loss value. Using this method one can show in just a couple of minutes that the  $n = 5$  game is a second player win. Also, the  $n = 6$  game can be reduced to a simpler game that we were able to analyze with a computer. Assuming the correctness of our program, we can assert that the  $n = 6$  game is also a second player win.

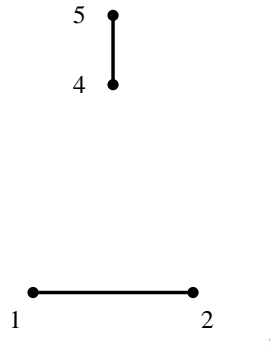
### 1. THE REDUCTION TECHNIQUE

We should first mention that while the game can be thought of geometrically, it is still a combinatorial game in the sense that the choice of triangulation matters.

For example, the simplicial complexes



are different triangulations of the same space, but the first is a first player win, while the second is a second player win. A winning move in the first game is to erase the vertex 3, leaving

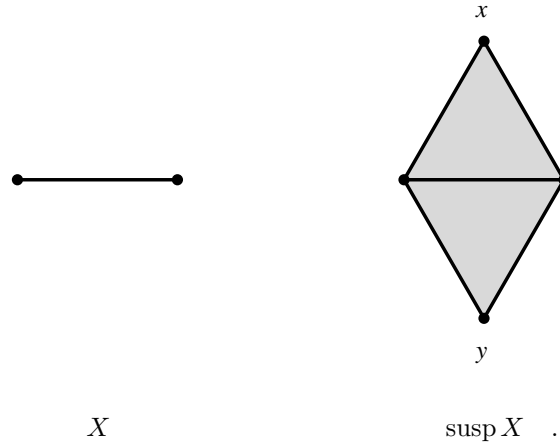


which is a second player win by symmetry. In the second game, the second player responds to the first according to the following strategy:

$$5 \longleftrightarrow 123 \quad 3 \longleftrightarrow 1 \text{ or } 2 \quad 12 \longleftrightarrow 35 \quad 13 \longleftrightarrow 23.$$

Now, despite the fact that the game seems to have little to do with topology, Keith Orpen suggested that we think about suspension since the conjecture deals with spheres, and the suspension of a  $k$ -sphere is a  $(k+1)$ -sphere. If  $X$  is a collection of subsets of  $A$  which form a simplicial complex, then the **suspension**  $\text{susp } X$  of  $X$  is the simplicial complex consisting of the following subsets of  $A \cup \{x, y\}$ , where  $x$  and  $y$  are new vertices:  $\text{susp } X$  contains the sets in  $X$ , the sets  $a \cup \{x\}$  and  $a \cup \{y\}$  for each set  $a$  in  $X$ , and the sets  $\{x\}$  and  $\{y\}$ . For example, the suspension of the

interval is a diamond:



It turns out that  $X$  and  $\text{susp } X$  always have the same win/loss value, and so a position of the form  $\text{susp } X$  can be *reduced*. However, if  $X$  is the  $k$ -sphere triangulated as the boundary of the  $(k + 1)$ -simplex, then  $\text{susp } X$  is topologically a  $(k + 1)$ -sphere but is *not* triangulated as the boundary of the  $(k + 2)$ -simplex. Therefore this reduction can't be used directly to attack the conjecture.

There is a more general type of reduction that is possible. We say that a pair  $(x, y)$  of vertices in a simplicial complex  $X$  is a **binary star** if

- there is no edge connecting  $x$  and  $y$  (*i.e.*,  $\{x, y\}$  is not present),
- for each set  $a$  in  $X$ , if  $a$  contains  $x$  then there is a set  $b$  which is the same as  $a$  except that it contains  $y$  in place of  $x$ , and
- the same as the previous item with  $x$  and  $y$  interchanged.

When this is the case, the vertices  $x$  and  $y$  can be removed (along with the simplices containing them) without changing who wins. Indeed, if player A can win in the reduced game, then this player can win in the larger game by using the following strategy: when B makes a move involving  $x$  or  $y$ , A makes the corresponding move with  $x$  and  $y$  interchanged; when B makes a move not involving  $x$  or  $y$ , A replies with the winning response in the reduced game.

Here's an example of the power of binary star reduction. Consider the starting position when  $A = \{1, 2, \dots, n\}$ . If an edge is removed, then its endpoints form a binary star in the resulting position. Performing reduction produces a (filled in)  $(n - 3)$ -simplex. It is easy to see that this position is a first player win, since the first player can "pass" by removing the interior if no other move wins, thereby leaving the second player with a losing position. Thus we have proved that choosing an edge from the starting position is a losing move.

Similarly, one can show that moves of size 1,  $n - 2$  and  $n - 1$  are all bad in the standard starting position of size  $n$ , assuming Gale's conjecture for smaller starting positions. (By a move of size 1 we mean a vertex, etc.) When  $n = 5$  these are the only moves available, so Gale's conjecture holds in this case. A reader who verified this without binary star reduction will realize how much effort we have saved. When  $n = 6$  we only have to analyze the situation after playing a move of size 3. Based on our experience with smaller games we guessed that the complementary set is a

winning response, and the computer verified this. In this way we verified Gale's conjecture for  $n = 6$  and were led to the following conjecture.

**Conjecture 2.** *A winning response to an opening move is to play the complementary move.*

2. A COUNTEREXAMPLE

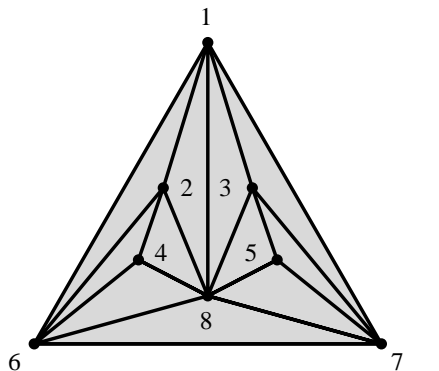
The game



is clearly a first player win. Also, each of the games homeomorphic to the 1-simplex, namely



is a first player win since the first player can choose the central simplex and then play by symmetry. One might hope that any triangulation of the  $k$ -simplex is a first player win. (Indeed, this implies Gale's conjecture.) However, the following is a counterexample:



This triangulation of the 2-simplex is a second player win. Notice that by adding the 2-simplex 167 to the above picture, one gets a triangulation of the 2-sphere which is a first player win. We wonder whether there are simpler triangulations of the 2-simplex and the 2-sphere that have these win/loss values.

REFERENCES

[1] Richard K. Guy. Unsolved problems in combinatorial games. *Games of No Chance*, (R. J. Nowakowski ed.) MSRI Publications **29**, Cambridge University Press, 1996, pp. 475–491.

**This version has been updated compared to the published version:**

The published version appeared in *American Mathematical Monthly* **104** (1997), 762–766. Since then, the following changes have been made:

- In the proof that choosing an edge is a losing move, “ $(n - 2)$ -simplex” has been corrected to “ $(n - 3)$ -simplex”.
- The reference to [1] has been updated.
- Minor formatting changes have been made.

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