DETECTING ISOMORPHISMS IN THE HOMOTOPY CATEGORY

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Abstract. We show that the homotopy category of unpointed spaces admits no set of objects jointly reflecting isomorphisms by giving an explicit counterexample involving large symmetric groups. We also show that, in contrast, the spheres jointly reflect equivalences in the homotopy 2-category of spaces. The non-existence of such a set in the homotopy category was originally claimed by Heller, but his argument relied on the statement that for every set of spaces, long enough transfinite sequential diagrams admit weak colimits which are privileged with respect to the given set. Using the theory of graphs of groups, we show that this statement is false, by proving that for every ordinal with uncountable cofinality, there is a diagram indexed by that ordinal which admits no weak colimit that is privileged with respect to the spheres.

1. Introduction

Let **Hot** denote the homotopy category of spaces, and let $\mathbf{Hot}_{*,c}$ denote the homotopy category of pointed, connected spaces. In [2], Brown proved that a functor $\mathbf{Hot}_{*,c}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact, in the sense that it sends coproducts and weak pushouts in $\mathbf{Hot}_{*,c}$ to products and weak pullbacks in **Set**. In [5], Heller proved an abstract representability theorem: if $\mathbf{C}$ is a category with coproducts and weak pushouts and $\mathbf{C}$ contains a bounded set $\mathcal{G}$ of objects that jointly reflects isomorphisms (see Definition 1.1 below), then a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact. In the same paper, Heller gave an example of a half-exact functor $\mathbf{Hot}^{\mathrm{op}} \to \mathbf{Set}$ which is not representable. He then claimed without proof [5, Prop. 1.2] that every set of spaces in **Hot** is bounded, and concluded [5, Cor. 2.3] that no set of spaces jointly reflects isomorphisms in **Hot**.

In this paper, we show that it is not true that every set of spaces is bounded, reopening the question of whether there is a set of spaces that jointly reflects isomorphisms in **Hot**. We then give an independent proof that no set of spaces jointly reflects isomorphisms.

We now give the definitions needed in order to precisely state our results.

Definition 1.1. Let $\mathbf{C}$ be any category and let $\mathcal{G} \subseteq \mathbf{C}$ be a set of objects.

1. We say that $\mathcal{G}$ *jointly reflects isomorphisms* if a morphism $f : X \to Y$ in $\mathbf{C}$ is an isomorphism whenever $\mathbf{C}(S, f) : \mathbf{C}(S, X) \to \mathbf{C}(S, Y)$ is a bijection for every $S \in \mathcal{G}$. (Heller uses the terminology "left adequate.")
2. A *weak colimit* of a diagram $D : \mathbf{I} \to \mathbf{C}$ is a cocone through which every cocone factors, not necessarily uniquely.

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(3) A weak colimit W of $D : \mathbf{I} \to \mathbf{C}$ is *$\mathcal{G}$-privileged* if the canonical map

$$\operatorname*{colim}_{\alpha \in \mathbf{I}} \mathbf{C}(S, D(\alpha)) \to \mathbf{C}(S, W)$$

is a bijection for every $S \in \mathcal{G}$.

(4) For an ordinal β, we say that $\mathcal{G}$ is *β-bounded* if every diagram $D : \beta \to \mathbf{C}$ has a $\mathcal{G}$-privileged weak colimit.

(5) We say that $\mathcal{G}$ is *left cardinally bounded*, or just *bounded*, if it is β-bounded for each sufficiently large regular cardinal β.

As mentioned above, **Hot** denotes the homotopy category of spaces, by which we mean the localization of the category of spaces at the weak homotopy equivalences, or equivalently, the category whose objects are CW-complexes and whose morphisms are homotopy classes of continuous maps. We use the word "set" to mean what is sometimes called a "small set," i.e., an object of the category **Set**. All of our ordinals and cardinals are "small."

We can now state our main results more precisely. First we give the result that shows that [5, Prop. 1.2] is false.

Theorem 3.1. *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in* **Hot** *is not κ-bounded for any ordinal κ of uncountable cofinality. That is, for each such κ, there exists a diagram $D : \kappa \to$* **Hot** *that admits no $\mathcal{G}$-privileged weak colimit.*

We immediately deduce:

Corollary 3.2. *Let T denote a countable, discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ-bounded in* **Hot** *for any limit ordinal κ.*

The proof of Theorem 3.1 is somewhat involved and forms the bulk of the paper. We first show that it is sufficient to find a counterexample in the homotopy category **HoGpd** of groupoids. Then, given κ as in the statement, we define a simple diagram $D : \kappa \to \mathbf{HoGpd}$. We make use of the theory of graphs of groups [7] and the associated fundamental groupoid [6] in order to construct a sufficiently pathological cocone $D \to Z$ which we use to show that D admits no $\mathcal{G}$-privileged weak colimit. This involves a detailed understanding of the morphisms in Z and how they are expressed as words in the given generators.

In the introduction to [3], Franke suggests an approach to showing that diagrams indexed by large ordinals may not admit $\mathcal{G}$-privileged weak colimits (for $\mathcal{G}$ a set of objects that jointly reflects isomorphisms) by comparing weak colimits to homotopy colimits. In order to complete the argument, it appears that one would need to show that a certain differential in a Bousfield-Kan spectral sequence is non-zero, and we were unable to find an example in which we could prove this. It does follow from our argument that the diagram we construct has a homotopy colimit which is not a weak colimit, as Franke suggested would be the case.

In the homotopy category of pointed, connected spaces, the set of spheres jointly reflects isomorphisms. However, we conjecture that the set of spheres is not bounded in $\mathbf{Hot}_{*,c}$. If this is true, it means that Heller's abstract representability theorem, as stated, does not imply Brown's representability theorem. That said, Heller's argument only requires a set of objects that jointly reflects isomorphisms and is β-bounded for *some* regular cardinal β. Thus, since the set of spheres is $\aleph_0$-bounded, the proof of Heller's theorem goes through in $\mathbf{Hot}_{*,c}$.

Next we state the result that shows that the statement of [5, Cor. 2.3] is nevertheless correct.

Theorem 2.1. *The category* **Hot** *contains no set $\mathcal{G}$ of spaces that jointly reflects isomorphisms. That is, there exists no set $\mathcal{G}$ of spaces such that, if $f : X \to Y$ is a map of spaces and $f_* : \mathbf{Hot}(S, X) \to \mathbf{Hot}(S, Y)$ is a bijection for every $S \in \mathcal{G}$, then f is an isomorphism in* **Hot**.

This second result is easier to prove, and so we prove it first, in Section 2. We generalize a well-known example of a "phantom homotopy equivalence," that is, a map in **Hot** which while not an isomorphism is seen as one by all finite complexes. Our proof also shows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces. Moreover, Theorem 2.1 implies similar results in other settings. For example, since **Hot** is a reflective subcategory of the homotopy category of $(\infty, 1)$-categories, it follows that there is no set of $(\infty, 1)$-categories that jointly reflects isomorphisms.

Since the $(\infty, 1)$-category $\mathcal{S}$ of spaces certainly contains a set of objects jointly reflecting equivalences—namely the point—while its 1-categorical truncation **Hot** does not, one might ask which behavior the n-categorical truncations of $\mathcal{S}$ exhibit. In fact, we show in Theorem 4.4 that in the 2-category $\underline{\mathbf{Hot}}$ of spaces, morphisms, and homotopy classes of homotopies between them, the set of spheres *does* jointly reflect equivalences, which is the natural generalization of joint reflection of isomorphisms to 2-category theory. Intuitively, the reason for the divergent behavior of **Hot** and $\underline{\mathbf{Hot}}$ is that the 2-morphisms of $\underline{\mathbf{Hot}}$ retain the information about based homotopies that is lost in **Hot**.

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2. **Hot** admits no set that jointly reflects isomorphisms

We make the following definitions. For an ordinal α, write Σ_α for the group of all bijections of the set α, ignoring order. When $\beta < \alpha$, there is a natural inclusion $\Sigma_\beta \hookrightarrow \Sigma_\alpha$, and we define $\Sigma^{\mathrm{c}}_\alpha$ to be the union of the images of Σ_β for all $\beta < \alpha$. We typically consider $\Sigma^{\mathrm{c}}_\alpha$ when α is a cardinal, considered as the smallest ordinal with that cardinality, and we call the elements of $\Sigma^{\mathrm{c}}_\alpha$ *essentially constant* permutations.

Theorem 2.1. *The category* **Hot** *contains no set $\mathcal{G}$ of spaces that jointly reflects isomorphisms. (See Definition 1.1.)*

Proof. Let $\mathcal{G}$ be a set of spaces and let α be a regular cardinal larger than the cardinality of $\pi_1(S, s_0)$ for each $S \in \mathcal{G}$ and each $s_0 \in S$. We must construct a map $f : X \to Y$ which is not a homotopy equivalence but which induces bijections on homotopy classes of maps from spaces in $\mathcal{G}$.

Our example will be $Bs : B\Sigma^{\mathrm{c}}_\alpha \to B\Sigma^{\mathrm{c}}_\alpha$, where $s : \Sigma^{\mathrm{c}}_\alpha \to \Sigma^{\mathrm{c}}_\alpha$ is the shift homomorphism given by

$$(s\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1, & \gamma = \gamma' + 1 \\ \gamma, & \gamma \text{ a limit ordinal,} \end{cases}$$

for $\sigma \in \Sigma^{\mathrm{c}}_\alpha$. (Here and in what follows, if γ is a successor ordinal, we write γ' for its predecessor.) We must check that $s\sigma \in \Sigma^{\mathrm{c}}_\alpha$. First, it is essentially constant: if $\beta < \alpha$ and σ fixes each $\gamma \geq \beta$, then for $\gamma > \beta$ we have $(s\sigma)(\gamma) = \gamma$, if γ is a limit ordinal, and $(s\sigma)(\gamma) = \sigma(\gamma') + 1 = \gamma' + 1 = \gamma$, if γ is a successor. Next, we see that s is a homomorphism:

$s(\sigma\tau)$ and $(s\sigma)(s\tau)$ both fix all limit ordinals, while for successors we have

$$(s\sigma)((s\tau)(\gamma)) = \sigma([\tau(\gamma')+1]')+1 = \sigma\tau(\gamma')+1 = s(\sigma\tau)(\gamma),$$

as desired. Note that setting $\tau = \sigma^{-1}$, respectively $\sigma = \tau^{-1}$, we confirm that $s\sigma$ is indeed a bijection.

Recall that for a group H and a connected space X, $\mathbf{Hot}(X, BH)$ is isomorphic to $\mathrm{Hom}(\pi_1(X), H)$ modulo conjugation by elements of H, where B denotes the classifying space functor. (See, for example, [8, Corollary V.4.4].) In particular, we have a natural isomorphism $\mathbf{Hot}(X, BH) \cong \mathbf{Hot}(B\pi_1(X), BH)$. It also follows that for groups G and H, $\mathbf{Hot}(BG, BH)$ is isomorphic to $\mathrm{Hom}(G, H)$ modulo conjugation by elements of H, and that an element of $\mathbf{Hot}(BG, BH)$ is a homotopy equivalence if and only if it is represented by an isomorphism.

Note that s is not surjective, since $s\sigma$ always preserves limit ordinals. Therefore, $Bs : B\Sigma^{\mathrm{c}}_\alpha \to B\Sigma^{\mathrm{c}}_\alpha$ is not a homotopy equivalence. However, we will show that it induces an isomorphism on $\mathcal{G}$. First observe that it suffices to prove this for connected components of spaces in $\mathcal{G}$. It follows that it is enough to prove this for spaces of the form BG, where G is a group of cardinality less than α.

Any map $BG \to B\Sigma^{\mathrm{c}}_\alpha$ arises from a homomorphism $\varphi : G \to \Sigma^{\mathrm{c}}_\alpha$, well-defined up to conjugation. Since α is regular, there is a limit ordinal $\beta < \alpha$ so that $\varphi(g) \in \Sigma_\beta$ for every $g \in G$. We claim that $s \circ \varphi$ is conjugate to φ by an element $\tau \in \Sigma^{\mathrm{c}}_\alpha$ defined as follows:

$$\tau(\gamma) = \begin{cases} \gamma', & \gamma < \beta \text{ a successor ordinal} \\ \beta+\gamma, & \gamma < \beta \text{ a limit ordinal} \\ \gamma+1, & \beta \le \gamma < \beta+\beta \\ \gamma, & \text{otherwise.} \end{cases}$$

It is straightforward to check that τ is a permutation, and it clearly fixes ordinals greater than or equal to $\beta+\beta$, which is less than α. For $g \in G$, let $\sigma = \varphi(g)$. Then, noting that $\tau^{-1}(\gamma) = \gamma + 1$ for any $\gamma < \beta$, we have

$$\begin{aligned}(\tau^{-1}\sigma\tau)(\gamma) &= \begin{cases} \tau^{-1}(\sigma(\gamma')), & \gamma < \beta \text{ a successor ordinal} \\ \tau^{-1}(\sigma(\beta+\gamma)), & \gamma < \beta \text{ a limit ordinal} \\ \tau^{-1}(\sigma(\gamma+1)), & \beta \le \gamma < \beta+\beta \\ \tau^{-1}(\sigma(\gamma)), & \text{otherwise} \end{cases} \\ &= \begin{cases} \tau^{-1}(\sigma(\gamma')), & \gamma < \beta \text{ a successor ordinal} \\ \tau^{-1}(\beta+\gamma), & \gamma < \beta \text{ a limit ordinal} \\ \tau^{-1}(\gamma+1), & \beta \le \gamma < \beta+\beta \\ \tau^{-1}(\gamma), & \text{otherwise} \end{cases} \\ &= \begin{cases} \sigma(\gamma')+1, & \gamma < \beta \text{ a successor ordinal} \\ \gamma, & \gamma < \beta \text{ a limit ordinal} \\ \gamma, & \beta \le \gamma < \beta+\beta \\ \gamma, & \text{otherwise} \end{cases} \\ &= s(\sigma)(\gamma).\end{aligned}$$

We have used that if $\gamma \ge \beta$, then $\sigma(\gamma) = \gamma$, and the consequence that if $\gamma < \beta$, then $\sigma(\gamma) < \beta$.

In summary, we have shown that Bs induces the identity on $\mathbf{Hot}(S, B\Sigma^{c}_{\alpha})$ for every $S \in \mathcal{G}$, proving the claim. □

Remark 2.2. Since the map $Bs : B\Sigma^{c}_{\alpha} \to B\Sigma^{c}_{\alpha}$ used in the proof has connected domain and codomain, it follows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces.

We end this section with a remark about the origin of the maps s and τ. Morally, s is conjugation by the successor operation on ordinals, with limit ordinals handled specially. The map τ implements this by "making room" for the relevant limit ordinals in a range outside of the support of a particular permutation σ. In fact, if we denote the map τ above by τ_{β}, then s itself is conjugation by τ_{α} in Σ^{c}_{γ} for a regular cardinal $\gamma > \alpha$.

3. The lack of privileged weak colimits

In this section, we give an example showing that Heller's privileged weak colimits do not generally exist.

Theorem 3.1. *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in* **Hot** *is not κ-bounded for any ordinal κ of uncountable cofinality, e.g., for any uncountable regular cardinal. That is, for each such κ, there exists a diagram $D : \kappa \to$* **Hot** *that admits no $\mathcal{G}$-privileged weak colimit.*

In particular, D admits no $\mathcal{G}$-privileged colimit for any set $\mathcal{G}$ containing the spheres. Note that the set of spheres is $\aleph_0$-bounded, so we learn that boundedness for one ordinal does not imply it for ordinals with larger cofinality.

Corollary 3.2. *Let T denote a countable, discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ-bounded in* **Hot** *for any limit ordinal κ.*

Proof. If κ has uncountable cofinality, then Theorem 3.1 applies. If κ has countable cofinality, then $\{T\}$ is not κ-bounded. □

In Section 3.1, we reduce the problem to finding a counterexample in the homotopy category of groupoids. In Section 3.2, we recall the theory of graphs of groups, and prove some general results about the word problem in the fundamental groupoid of a graph of groups. Finally, in Section 3.3, we give a counterexample in the homotopy category of groupoids and complete the proof of Theorem 3.1.

3.1. Reducing from spaces to groupoids.

To prove Theorem 3.1 we will work primarily in the homotopy category **HoGpd** of groupoids, that is, the category of groupoids and isomorphism classes of functors. It is well known that the geometric realization of groupoids induces a reflective embedding $B :$ **HoGpd** $\to$ **Hot** whose left adjoint is the fundamental groupoid functor Π_1. This follows from the adjunction between π_1 and the classifying space functor B that was used in the proof of Theorem 2.1.

Lemma 3.3. *Suppose given a diagram $D : J \to$* **HoGpd***, a set $\mathcal{G}'$ of groupoids, and a set $\mathcal{G}$ of spaces containing $B(\mathcal{G}')$ as well as S^n for all n. If D admits no $\mathcal{G}'$-privileged weak colimit in* **HoGpd***, then $B \circ D : J \to$* **Hot** *admits no $\mathcal{G}$-privileged weak colimit in* **Hot***.*

Proof. We prove the contrapositive. Let $\lambda : B \circ D \to X$ be a $\mathcal{G}$-privileged weak colimit, with $X \in$ **Hot**. Then, since left adjoints preserve weak colimits, $\Pi_1(\lambda) : D \to \Pi_1 X$ is a weak colimit. We will show that it is $\mathcal{G}'$-privileged.

First, since λ is $\mathcal{G}$-privileged, every map $a : S^n \to X$ factors through a 1-type $B(D_j)$ for some j. Thus, when $n > 1$, a is freely homotopic to a constant, which implies that $\pi_n(X, x)$ is trivial for all $x \in X$. We conclude that X is a 1-type itself, so that $X \simeq B(\Pi_1 X)$.

Since B is fully faithful, we see that $\Pi_1(\lambda) : D \to \Pi_1 X$ is $\mathcal{G}'$-privileged. Indeed, if $G \in \mathcal{G}'$, then

$$\begin{aligned}\mathbf{HoGpd}(G, \Pi_1 X) &\cong \mathbf{Hot}(B(G), B(\Pi_1 X)) \cong \mathbf{Hot}(B(G), X) \\ &\cong \operatorname*{colim}_j \mathbf{Hot}(B(G), B(D_j)) \cong \operatorname*{colim}_j \mathbf{HoGpd}(G, D_j).\end{aligned}$$

One can show that the composite isomorphism is induced by $\Pi_1(\lambda)$. □

Thus it suffices to exhibit appropriately pathological diagrams in **HoGpd**, and then to upgrade them to **Hot**. We aim to give a diagram in **HoGpd** admitting no weak colimit privileged with respect to the set $\mathcal{G}' = \{\mathbf{B}\mathbb{Z}\}$. Here $\mathbf{B}\mathbb{Z}$ denotes the groupoid freely generated by an automorphism, i.e., the groupoid with one object $*$ whose endomorphism group is the integers. Of course, $B(\mathbf{B}\mathbb{Z})$ is homotopy equivalent to S^1, so $\mathcal{G}$ in Lemma 3.3 can be taken to be the set of spheres.

Remark 3.4. Note that, for any groupoid G, a functor $f : \mathbf{B}\mathbb{Z} \to G$ corresponds to an object $f(*)$ of G and an automorphism $f_* : f(*) \to f(*)$. Furthermore, two such functors $f, g : \mathbf{B}\mathbb{Z} \to G$ are naturally isomorphic if and only if the automorphisms f_* and g_* are conjugate in G. In particular, a functor $f : \mathbf{B}\mathbb{Z} \to G$ factors through $h : H \to G$ in **HoGpd** if and only if f_* is conjugate to an automorphism in the image of h.

3.2. **Graphs of groups.** To construct our example, we recall the notion of a graph of groups, and prove Corollaries 3.7, 3.8 and 3.9 that will be used in the next section.

Definition 3.5. A *graph of groups* Γ is given by:

- A graph, i.e., a set X of vertices, a set Y of oriented edges, functions $s, t : Y \rightrightarrows X$, and an involution $\overline{(-)} : Y \to Y$ interchanging s and t.
- Groups G_x and G_y for $x \in X$ and $y \in Y$ equipped with monomorphisms $\mu_y : G_y \to G_{s(y)}$ such that $G_y = G_{\bar{y}}$.

For simplicity, we assume that the groups G_x are disjoint. For more on graphs of groups, see [7, Section I.5] and [4, Section 1.B].

Higgins [6] defined the fundamental groupoid $\Pi_1\Gamma$ of a graph of groups. The groupoid $\Pi_1\Gamma$ is the groupoid on objects X with generating morphisms the elements of the groups G_x, endowed with x as domain and codomain, together with the elements of Y viewed as morphisms $y : s(y) \to t(y)$. These generators are subject to the relations holding in the groups G_x, as well as new relations

$$\mu_{\bar{y}}(a) = y\mu_y(a)\bar{y},$$

for every y and every $a \in G_y$. Note in particular that $\bar{y} = y^{-1}$, and we shall use both notations. It may aid the intuition to consider $\Pi_1\Gamma$ as the fundamental groupoid of the space built from $\coprod_X BG_x$ with cylinders $BG_y \times I$ glued in for each set $\{y, \bar{y}\}$ of elements of Y related by the involution.

By definition, the groupoid $\Pi_1\Gamma$ is a quotient of the groupoid $\mathbf{K}$ with object set X and with morphisms freely generated by $(\coprod G_x) \coprod Y$, subject to the relations holding in the groups G_x. A morphism $x_0 \to x_n$ in $\mathbf{K}$ is given by a word $(a_n, y_n, \ldots, y_1, a_0)$, with $y_i \in Y$, $s(y_1) = x_0$, $t(y_n) = x_n$, and $s(y_{i+1}) = t(y_i) =: x_i$ for $1 \le i < n$, while $a_i \in G_{x_i}$ for $0 \le i \le n$.

The natural realization functor $\mathbf{K} \to \Pi_1\Gamma$ will be denoted by $|(a_n, y_n, \ldots, y_1, a_0)| = a_n \circ y_n \circ \cdots \circ y_1 \circ a_0$. Higgins proves that every morphism of $\Pi_1\Gamma$ is *uniquely* the image under $|\cdot|$ of a so-called "normal" word. We will not recall this concept, as we need only Higgins' corollary regarding the less rigid *irreducible* words.

A morphism $(a_n, y_n, \ldots, y_1, a_0)$ in $\mathbf{K}$ is called *reducible* if $n > 1$ and for some i, $y_{i-1} = \bar{y}_i$ and $a_{i-1} \in \mu_{y_i}(G_{y_i})$. Otherwise, the morphism is said to be *irreducible*. Note that a reducible word can be shortened by the move

$$(\ldots, a_i, y_i, \mu_{y_i}(\hat{a}_{i-1}), \bar{y}_i, a_{i-2}, \ldots) \mapsto (\ldots, a_i\mu_{\bar{y}_i}(\hat{a}_{i-1})a_{i-2}, \ldots)$$

to a word with the same realization. Therefore, every element of $\Pi_1\Gamma$ is the realization of an irreducible word. We will use a key result of [6].

Proposition 3.6 ([6, Corollary 5])**.** *Let w be an irreducible word in $\mathbf{K}$. If $|w|$ is an identity morphism in $\Pi_1\Gamma$, then $w = (e)$, where e is an identity element of some G_x.*

Define the *length* $\ell(w)$ of the word $w = (a_n, y_n, \ldots, y_1, a_0)$ to be n. We deduce the following:

Corollary 3.7. *Let Γ be a graph of groups and consider a word w in the groupoid $\mathbf{K}$. If $\ell(w) > 0$ and $|w|$ is equal to the realization of a zero-length word, then w is reducible.*

Proof. Suppose that $w = (a_n, y_n, \ldots, y_1, a_0)$ for $n > 0$ and that $|w| = |(a)|$ for some a in some G_x. Let $w' = (a_n, y_n, \ldots, y_1, a_0\, a^{-1})$. Then $|w'|$ is an identity morphism in $\Pi_1\Gamma$, so by Proposition 3.6, w' is reducible. Since reduction occurs at interior points, w must be reducible as well. □

Corollary 3.8. *Given a graph of groups Γ and a vertex x, the vertex group G_x embeds in the automorphism group of x in the fundamental groupoid $\Pi_1\Gamma$.*

Because of this, we regard elements of the vertex groups as elements of the fundamental groupoid without explicitly naming the inclusion map.

Proof. The map sends $a \in G_x$ to the realization of the word (a). Since the word (a) is irreducible, if the realization is an identity in $\Pi_1\Gamma$, Proposition 3.6 tells us that a is the identity element of G_x. Therefore, this map is injective. □

We record some facts about free groups, which are the fundamental groupoids of graphs of groups with $X = \{x\}$ a singleton and G_x trivial.

Corollary 3.9. *Let $A \subseteq B$ be nonabelian free groups, with A free on generators $\{a_i\}$ and B free on $\{a_i\} \cup \{b_j\}$.*

1. *If $b \in B$ and for all $a \in A$ we have $bab^{-1} = a$, then b is the identity.*
2. *If $b \in B$ satisfies $bab^{-1} \in A$ for some $a \in A$, then either a is the identity or $b \in A$.*

Proof. Fix $b \in B$. For part (1), if we take $a = a_i$ then the assumption that $ba_ib^{-1} = a_i$ shows that an irreducible word for b must have last letter a_i for every i, which is absurd since there are at least two i's.

For part (2), we assume a is nontrivial and $b \notin A$. Factor b as $b'b''$, where $b'' \in A$ while b' is represented by an irreducible word with rightmost letter some b_j. Then $bab^{-1} = b'a'b'^{-1}$, where $a' := b''a\,b''^{-1}$ is a non-trivial element of A. The conclusion now follows from the observation that no reductions are possible in the concatenation of the irreducible words for b', a' and b'^{-1}, since concatenating those words gives no letter adjacent to its inverse. □

3.3. **A counterexample in the homotopy category of groupoids.** We now apply the generalities above to the problem of weak colimits in **HoGpd**.

We fix for the rest of the paper an ordinal κ of uncountable cofinality, and introduce the main characters in our counterexample. Note that Theorem 3.1 will follow if we replace $\kappa = [0, \kappa)$ by the interval $[2, \kappa)$, since the two categories are isomorphic.

Definition 3.10. Define a graph of groups Γ with object set $[2, \kappa)$, vertex group G_α free on α generators, edge set $\{y_\alpha^\beta : \beta \to \alpha \mid \alpha \neq \beta \in [2, \kappa)\}$, and involution $y_\alpha^\beta \mapsto y_\beta^\alpha$. The edge group $G_{y_\alpha^\beta}$ is just $G_{\min(\beta,\alpha)}$. The edge morphism $\mu_{y_\alpha^\beta} : G_{\min(\beta,\alpha)} \to G_\beta$ is the natural inclusion. Let $Z = \Pi_1 \Gamma$.

Next, define a diagram $D : [2, \kappa) \to \mathbf{HoGpd}$ by letting $D(\alpha)$ be free on α generators with action on morphisms the natural inclusions, denoted by $D_\alpha^\beta : D(\beta) \to D(\alpha)$. We have a cocone $A : D \to Z$ with $A_\alpha : D(\alpha) \to Z$ the natural inclusion of the vertex group. To see that these maps do constitute a cocone, we note that y_α^β is the unique component of a natural isomorphism $A_\beta \cong A_\alpha \circ D_\alpha^\beta$.

Critically, we do *not* have the relations $y_\alpha^\beta y_\beta^\gamma = y_\alpha^\gamma$ in Z which would allow us to lift A into a cocone in the 2-category of groupoids. We now intend to show that D admits no privileged weak colimit by, roughly, showing that this failure is unavoidable: no choice of isomorphisms $A_\beta \cong A_\alpha \circ D_\alpha^\beta$ can give A such a lift.

Write Z_Y for the subgroupoid of Z generated by the edges of the graph. Any morphism of Z_Y can be uniquely written as a reduced word in the generators y_α^β. We say that such a morphism *passes through* a vertex α if this unique word involves a generator with source or target α. The identity id_α is said to *pass through* α and no other vertex.

Lemma 3.11. *Let $u : \beta \to \alpha$ in Z and let $2 \leq \gamma \leq \min(\alpha, \beta)$. Then u is in Z_Y and does not pass through any vertex less than γ if and only if u is the unique component of a natural isomorphism $A_\alpha \circ D_\alpha^\gamma \cong A_\beta \circ D_\beta^\gamma$. That is, for all $a \in G_\gamma$, we must have $D_\alpha^\gamma(a) = u D_\beta^\gamma(a) u^{-1}$ in Z.*

Proof. Suppose that u is in Z_Y and does not pass through any vertex less than γ. It suffices to show that y_α^β conjugates D_β^γ into D_α^γ when $\gamma \leq \beta \leq \alpha$. In this case, $\mu_{y_\alpha^\beta}$ is an identity map, and so the claim follows from the defining relations of Z:

$$y_\alpha^\beta\, D_\beta^\gamma(a)\, \bar{y}_\alpha^\beta = y_\alpha^\beta\, \mu_{y_\alpha^\beta}(D_\beta^\gamma(a))\, \bar{y}_\alpha^\beta = \mu_{\bar{y}_\alpha^\beta}(D_\beta^\gamma(a)) = D_\alpha^\beta(D_\beta^\gamma(a)) = D_\alpha^\gamma(a).$$

For the converse, let u be the realization of an irreducible word $w = (a_n, y_n, \ldots, y_1, a_0)$. We proceed by induction on n. If $n = 0$, then $\alpha = \beta$ and $u = |(a_0)| \in G_\beta$. The assumption that $D_\beta^\gamma(a) = u D_\beta^\gamma(a) u^{-1}$ shows that u centralizes a nonabelian subgroup of a free group. By Corollary 3.9 (1), we see that u is trivial as desired. And clearly u does not pass through a vertex less than γ; indeed, it passes through only β, and $\beta \geq \gamma$.

For the inductive step, assume $n > 0$. Then $s(y_1) = \beta$ and $t(y_n) = \alpha$. Let $t(y_1) = \delta$, and note that $\delta \neq \beta$. In terms of w, the assumption on u is that the word

$$w' = (a_n, y_n, \ldots, y_1, a_0 D_\beta^\gamma(a) a_0^{-1}, y_1^{-1}, a_1^{-1}, \ldots, y_n^{-1}, a_n^{-1})$$

has realization $D_\alpha^\gamma(a)$ for every $a \in G_\gamma$. Thus, by Corollary 3.7, w' is reducible. Since by assumption w is irreducible, any reduction must occur at the central entry. So, letting $\varepsilon := \min(\beta, \delta)$, we must have $a_0 D_\beta^\gamma(a) a_0^{-1} \in \mu_{y_1}(G_\varepsilon) = D_\beta^\varepsilon(G_\varepsilon)$. In particular, $a_0 D_\beta^\varepsilon(a) a_0^{-1} \in D_\beta^\varepsilon(G_\varepsilon)$ for some non-identity element a in $G_{\min(\gamma,\varepsilon)}$. So by Corollary 3.9 (2), we see that

$a_0 = D^{\varepsilon}_{\beta}(\hat{a}_0)$ for some $\hat{a}_0 \in G_\varepsilon$. It then follows that $D^{\gamma}_{\beta}(a)$ is in the image of D^{ε}_{β} for every $a \in G_\gamma$, which means that $\gamma \leq \varepsilon$. The reduction of w at its central entry is

$$(a_n, y_n, \ldots, y_2, a_1\, D^{\varepsilon}_{\delta}(\hat{a}_0)\, D^{\gamma}_{\delta}(a)\, D^{\varepsilon}_{\delta}(a_0)^{-1}\, a_1^{-1}, y_2^{-1}, a_2^{-1}, \ldots, a_n^{-1}).$$

Thus, if we define $u' : \delta \to \alpha$ to be $|w''|$, where $w'' = (a_n, y_n, \ldots, y_2, a_1 D^{\varepsilon}_{\delta}(\hat{a}_0))$, then $\ell(w'') < n$ and u' conjugates D^{γ}_{δ} to D^{γ}_{α}. By induction, $u' \in Z_Y$. Since

$$u' y_1 = a_n y_n \cdots y_2 a_1 D^{\varepsilon}_{\delta}(\hat{a}_0) y_1 = a_n y_n \cdots y_2 a_1 y_1 D^{\varepsilon}_{\beta}(\hat{a}_0) = u,$$

u is in Z_Y as well. Finally, recall that we observed that $\gamma \leq \varepsilon = \min(\beta, \delta)$. By induction, u' does not pass through any vertex less than γ. So the same is true of $u = u' y_1$. □

Let Z_X denote the subgroupoid of Z containing those morphisms in the image of G_x for some x. By Corollary 3.8, Z_X is isomorphic to the disjoint union of the groups G_x.

Lemma 3.12. *Consider a morphism $z : \alpha \to \alpha$ in Z. If there are morphisms $u : \alpha \to \beta$ and $v : \alpha \to \gamma$ in Z such that uzu^{-1} is in Z_X and vzv^{-1} is in Z_Y, then $z = \mathrm{id}_\alpha$.*

Proof. Let $y = vzv^{-1}$. Note that the inclusion $Z_Y \to Z$ has a retraction $r : Z \to Z_Y$ defined by sending the generators of each vertex group to identity elements. Since $uv^{-1}yvu^{-1}$ is in Z_X, we have that $r(uv^{-1}yvu^{-1}) = r(uv^{-1})\, y\, r(uv^{-1})^{-1}$ is an identity, and so y is an identity. Since $y = vzv^{-1}$ is an identity, we have that z is an identity as well. □

The following is the key technical result.

Lemma 3.13. *Suppose given a family $u^{\beta}_{\alpha} : \beta \to \alpha$ of morphisms of Z_Y for all $\beta < \alpha \in [2, \kappa)$ such that $u^{\gamma}_{\alpha} = u^{\beta}_{\alpha} u^{\gamma}_{\beta}$ for all triples $\gamma < \beta < \alpha$. Then there exists a pair $\beta < \alpha$ such that u^{β}_{α} passes through some γ with $\gamma < \beta$.*

Proof. Assume that this is not the case. Let $\delta_0 = 2$ and $\delta_1 = 3$. Inductively, for each $n \in \omega$ let δ_n be an ordinal exceeding every vertex that $u^{\delta_{n-2}}_{\delta_{n-1}}$ passes through. This is possible because κ is a limit ordinal.

For each n, $u^{\delta_{n-1}}_{\delta_n}$ can be written uniquely as a reduced word in the free groupoid Z_Y. Let y_n be a letter in this word which is of the form y^{β}_{α} with $\beta < \delta_n \leq \alpha$. Such a letter must exist since $u^{\delta_{n-1}}_{\delta_n}$ starts at a vertex less than δ_n and ends at δ_n. Note that y_n cannot occur in the reduced form of any $u^{\delta_{k-1}}_{\delta_k}$ with $k \neq n$. For $k < n$, this holds by definition of δ_n, and for $k > n$, this holds by our assumption that each u^{β}_{α} only passes through γ with $\gamma \geq \beta$. In particular, the y_n's are distinct.

Using that κ has uncountable cofinality, choose $\delta_\omega < \kappa$ to be an ordinal exceeding every δ_n. Consider the decompositions

$$u^{\delta_0}_{\delta_\omega} = u^{\delta_1}_{\delta_\omega} u^{\delta_0}_{\delta_1} = u^{\delta_2}_{\delta_\omega} u^{\delta_1}_{\delta_2} u^{\delta_0}_{\delta_1} = u^{\delta_3}_{\delta_\omega} u^{\delta_2}_{\delta_3} u^{\delta_1}_{\delta_2} u^{\delta_0}_{\delta_1} = \cdots$$

In the expression $u^{\delta_1}_{\delta_\omega} u^{\delta_0}_{\delta_1}$, a y_1 occurs in the reduced form of the right-hand factor, and does not occur in the left-hand factor, so the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain a y_1. Similarly, the second decomposition involves a y_2, which can't be cancelled from either side, so the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain a y_2. Continuing, we see that the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain countably many distinct letters, a contradiction. □

Recall that κ is an arbitrary ordinal of uncountable cofinality.

Proposition 3.14. *There exists a diagram $C : [2, \kappa) \to$ **HoGpd** valued in the homotopy category of groupoids such that for any weak colimit with cocone $F : C \to W$, there exists an automorphism in W which is not conjugate to any morphism in the image of any leg $F_\alpha : C(\alpha) \to W$ of F.*

Proof. We claim that the diagram D (see Definition 3.10) is an example of such a C.

Towards a contradiction, suppose $F : D \to W$ is a weakly colimiting cocone such that every automorphism in W is conjugate to one in the image of some component of F. Write F_α for functors representing the maps $D(\alpha) \to W$. Since F is a cocone in **HoGpd**, for each $\beta < \alpha \in [2, \kappa)$ we may choose a natural isomorphism

$$h^\beta_\alpha : F_\beta \cong F_\alpha \circ D^\beta_\alpha$$

between functors $D(\beta) \to W$ in **Gpd**. Denote by $\hat{h}^\beta_\alpha$ the unique component of h^β_α. As usual we shall denote $(h^\beta_a)^{-1}$ by h^α_β, and similarly for $\hat{h}$, as well as u below.

Recall the natural cocone $A : D \to Z$ from Definition 3.10 and suppose given a representative $f : W \to Z$ of a factorization of the cocone A through F. For each α, pick a natural isomorphism $k_\alpha : A_\alpha \cong f \circ F_\alpha$ with unique component $\hat{k}_\alpha$. For $\beta < \alpha$, let $u^\beta_\alpha = \hat{k}^{-1}_\alpha f(\hat{h}^\beta_\alpha)\hat{k}_\beta$, the unique component of the natural transformation $A_\beta \to A_\alpha \circ D^\beta_\alpha$ defined by $(k^{-1}_\alpha * D^\beta_\alpha) \circ (f * h^\beta_\alpha) \circ k_\beta$, where $*$ denotes whiskering.[2] By Lemma 3.11, we see that each $u^\beta_\alpha \in Z_Y$, so the same holds for the morphism $u_{\alpha\beta\gamma} : \gamma \to \gamma$ defined as $u^\alpha_\gamma u^\beta_\alpha u^\gamma_\beta$ for $\gamma < \beta < \alpha$. Furthermore, the same lemma guarantees that no u^β_α passes through a vertex less than $\min(\beta, \alpha)$.

For each $\gamma < \beta < \alpha$, denote by $w_{\alpha\beta\gamma} \in W$ the unique component of the composite natural transformation

$$h^\alpha_\gamma \circ (h^\beta_\alpha * D^\gamma_\beta) \circ h^\gamma_\beta : F_\gamma \to F_\gamma.$$

We have $w_{\alpha\beta\gamma} = \hat{h}^\alpha_\gamma \hat{h}^\beta_\alpha \hat{h}^\gamma_\beta$, so

$$\hat{k}^{-1}_\gamma f(w_{\alpha\beta\gamma})\hat{k}_\gamma = \hat{k}^{-1}_\gamma f(\hat{h}^\alpha_\gamma)\hat{k}_\alpha \hat{k}^{-1}_\alpha f(\hat{h}^\beta_\alpha)\hat{k}_\beta \hat{k}^{-1}_\beta f(\hat{h}^\gamma_\beta)\hat{k}_\gamma = u_{\alpha\beta\gamma}.$$

In particular, $u_{\alpha\beta\gamma}$ is conjugate to $f(w_{\alpha\beta\gamma})$.

On the other hand, by assumption on F, $w_{\alpha\beta\gamma}$ is conjugate to a morphism in the image of some $F_\theta : D(\theta) \to W$, say to $F_\theta(w'_{\alpha\beta\gamma})$. Composing with f, we see that $u_{\alpha\beta\gamma}$ is conjugate to $f(F_\theta(w'_{\alpha\beta\gamma}))$. Finally, using $\hat{k}_\gamma$, we see $u_{\alpha\beta\gamma}$ is conjugate to $A_\theta(w'_{\alpha\beta\gamma})$, in particular, to an element of Z_X. Since we saw above that $u_{\alpha\beta\gamma}$ is in Z_Y, Lemma 3.12 shows that $u_{\alpha\beta\gamma} = \mathrm{id}_\gamma$.

Finally, Lemma 3.13 implies that at least one u^β_α passes through a vertex less than β, contradicting what we saw above. □

Proof of Theorem 3.1. By Proposition 3.14 and Remark 3.4, the diagram D admits no weak colimit privileged with respect to the set $\mathcal{G}' = \{\mathbf{B}\mathbb{Z}\}$. Thus by Lemma 3.3, $B \circ D$ admits no weak colimit in **Hot** which is privileged with respect to the set of spheres. □

4. The spheres reflect equivalences in the 2-category of spaces

We saw in Theorem 2.1 that in the homotopy category of spaces there is no set of objects that jointly reflects isomorphisms. In this section, we show that in the homotopy *2-category* of spaces, the spheres do jointly reflect equivalences. We first define the terms we are using.

[2]For instance, $f * h^\beta_\alpha : f \circ F_\beta \cong f \circ F_\alpha \circ D^\beta_\alpha$ has unique component $f(\hat{h}^\beta_\alpha)$.

Definition 4.1. By $\underline{\mathbf{Hot}}$, we mean the 2-category whose objects are spaces of the homotopy type of a CW-complex and whose hom-categories are the fundamental groupoids of mapping spaces, that is $\underline{\mathbf{Hot}}(X, Y) = \Pi_1(Y^X)$.

Definition 4.2. A set $\mathcal{G}$ of objects in a 2-category $\mathcal{K}$ *jointly reflects equivalences* if, whenever $f : X \to Y$ is a morphism in $\mathcal{K}$ such that, for every $S \in \mathcal{G}$, the induced functor $\mathcal{K}(S, f) : \mathcal{K}(S, X) \to \mathcal{K}(S, Y)$ is an equivalence of categories, then f itself must be an equivalence in $\mathcal{K}$.

We shall show in Theorem 4.4 that the 2-category $\underline{\mathbf{Hot}}$ admits a set $\mathcal{G}$ of objects that jointly reflects equivalences, namely $\mathcal{G} = \{S^n \mid n \geq 0\}$. Note that a map f in $\underline{\mathbf{Hot}}$ is an equivalence if and only if it is a homotopy equivalence.

First, we shall compute some homotopy groups of the unpointed mapping spaces X^{S^k}. We denote the constant map $S^k \to X$ valued at x by x. Then we have:

Lemma 4.3. *Let X be a space and let $x \in X$. Then for $n \geq 0$ and $k \geq 1$, the homotopy group $\pi_k(X^{S^n}, x)$ is given by a semidirect product $\pi_k(X, x) \ltimes \pi_{n+k}(X, x)$.*

Of course, the semidirect product is direct when $k > 1$.

Proof. The map $e : X^{S^n} \to X$ given by evaluation at a fixed point $*$ is a fibration with fiber $\Omega^n(X, x)$, the space of based maps $(S^n, *) \to (X, x)$. Furthermore, e is a split epimorphism, with splitting the map $X \to X^{S^n}$ assigning to $x \in X$ the constant map valued at x.

Thus if we consider X, X^{S^n}, and $\Omega^n(X, x)$ to be pointed by (the constant map valued at) x, the long exact sequence in homotopy groups determined by e degenerates into split short exact sequences

$$1 \to \pi_k\Omega^n(X, x) \to \pi_k(X^{S^n}, x) \to \pi_k(X, x) \to 1.$$

Since $\pi_k\Omega^n(X, x)$ is naturally identified with $\pi_{n+k}(X, x)$, the result follows. □

With this, we are prepared to show that the spheres satisfy the analogue of Whitehead's theorem for $\underline{\mathbf{Hot}}$.

Theorem 4.4. *The set $\mathcal{G} = \{S^n\}$ of spheres jointly reflects equivalences in the 2-category $\underline{\mathbf{Hot}}$ of spaces.*

Proof. Let $f : X \to Y$ be such that $\underline{\mathbf{Hot}}(S^n, f) : \underline{\mathbf{Hot}}(S^n, X) \to \underline{\mathbf{Hot}}(S^n, Y)$ is an equivalence of groupoids, for every n. First observe that since f induces an equivalence $\underline{\mathbf{Hot}}(S^0, X) \to \underline{\mathbf{Hot}}(S^0, Y)$, it also induces an equivalence $\underline{\mathbf{Hot}}(*, X) \to \underline{\mathbf{Hot}}(*, Y)$, that is to say, an equivalence $\Pi_1(X) \to \Pi_1(Y)$ of fundamental groupoids. Thus f induces an isomorphism on π_0 and on every π_1. We next show that it induces an isomorphism on every π_n.

We have by assumption that f induces isomorphisms

$$\underline{\mathbf{Hot}}(S^n, X)(x, x) \to \underline{\mathbf{Hot}}(S^n, Y)(f(x), f(x))$$

for every $x \in X$, where again x denotes the constant map valued at x. In other words, f induces isomorphisms $\pi_1(X^{S^n}, x) \to \pi_1(Y^{S^n}, f(x))$ for each $x \in X$. The action of f on these free loop spaces by postcomposition respects the fiber sequence used in Lemma 4.3 and so induces a map from the short exact sequence

$$1 \to \pi_{n+1}(X, x) \to \pi_1(X^{S^n}, x) \to \pi_1(X, x) \to 1$$

for X to the analogous sequence for Y. Since f induces isomorphisms on the central groups and the cokernels, it induces isomorphisms of π_n as well. So, by the classical form of Whitehead's theorem, f is a homotopy equivalence, and thus an equivalence in $\underline{\mathbf{Hot}}$. □

Remark 4.5. In fact, for any based cofibrant space (A, a), the map $e : X^A \to X$ from the space of unbased maps given by evaluation at a has the same properties as the evaluation map $X^{S^n} \to X$. That is, e is a fibration which admits a splitting by constant maps and whose fiber is the space $(X, x)^{(A,a)}$ of based maps $(A, a) \to (X, x)$. Thus $\pi_n(X^A, x)$ is identified with a (semi)direct product of $\pi_n(X, x)$ and $\pi_n((X, x)^{(A,a)}, x)$. When the latter is understood, we get a recipe for producing sets reflecting equivalences in $\underline{\mathbf{Hot}}$. For example, in [1], the first author used this method to show that the set $\{(S^1)^n\}$ of finite-dimensional tori also reflects equivalences in $\underline{\mathbf{Hot}}$, while the observation that it could also apply to the spheres is due to Raptis.

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