Non-accessible localizations

Dan Christensen
University of Western Ontario

HoTT/UF 2021, July 18, 2021

Outline:

- Motivation and history
- Smallness
- Localization at a large family

Slides, abstract and formalization:

`http://jdc.math.uwo.ca/papers/`

Motivation and history

Localizations have been used in topology since the work of Sullivan and Bousfield in the 1970s.

They have played a fundamental organizing role in the subject, influencing and leading to the solution of many central conjectures.

For example, given a space X and a prime p, there is an associated space $L_p X$ called the localization of X at p.

Studying such p-local spaces is easier than studying general spaces, and there are fracture theorems that can be used to reconstruct a space from its p-localizations and its rationalization.

Localizations now also play an important role in the theory of ∞-categories.

Modalities in logic associate to a proposition P a new proposition $\lozenge P$.

Reflective subuniverses

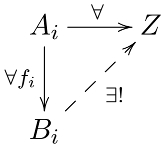
Rijke-Shulman-Spitters (2017)

A reflective subuniverse L of a universe $\mathcal{U}$ consists of:

- a subuniverse $\mathsf{is\text{-}local}_L : \mathcal{U} \to \mathsf{Prop}_{\mathcal{U}}$,
- a function $L : \mathcal{U} \to \mathcal{U}$,
- and a localization $\eta_X : X \to LX$ for each $X : \mathcal{U}$.

Being a localization means that LX is L-local and η_X is initial among maps whose codomain is local.

Given a type I and maps $f_i : A_i \to B_i$, for $i : I$, we say that a type Z is f-local if, for each i, every map $A_i \to Z$ extends uniquely to B_i:



Ex. Let $f : S^{n+1} \to 1$. The f-local types are exactly the n-types.

Reflective subuniverses II

Rijke-Shulman-Spitters (2017)

A reflective subuniverse L consists of $\mathsf{is\text{-}local}_L : \mathcal{U} \to \mathsf{Prop}_{\mathcal{U}}$, $L : \mathcal{U} \to \mathcal{U}$ and $\eta_X : X \to LX$ for each $X : \mathcal{U}$, as above.

Given $f : \prod_i (A_i \to B_i)$, Z is f-local if $f_i^* : Z^{B_i} \to Z^{A_i}$ is an equivalence for all i.

Theorem (RSS). If $I : \mathcal{U}$ and all $A_i, B_i : \mathcal{U}$, then the f-local types form a reflective subuniverse denoted L_f.

A reflective subuniverse that is presented as the f-local types for some family f in $\mathcal{U}$ is called accessible.

Question. Is every reflective subuniverse accessible?

Theorem (Casacuberta-Scevenels-Smith). In Spaces, the answer is independent of ZFC.

Accessibility and cardinals

CSS (2005)

Say that a space X is CSS-local if it is a 1-type and

$$\mathrm{Hom}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \pi_1(X,x)) = 1$$

for every $x \in X$ and every cardinal κ.

Theorem (Casacuberta-Scevenels-Smith).

(1) There is a localization onto the CSS-local types.

(2) If this localization is accessible, then a measurable cardinal exists.

(3) If Vopěnka's principle holds, then every localization is accessible.

The goal of the talk is to present a type-theoretic approach to (1), which also generalizes it.

Smallness

Let $\mathcal{U}$ be a univalent universe. We're using book HoTT.

A type X is small if $\sum_{X':\mathcal{U}} X' \simeq X$.

A type X is 0-locally small if it is small.
It is $(n+1)$-locally small if $x = x'$ is n-locally small for all $x, x' : X$.

Theorem (Rijke). If A is small, X is 1-locally small and $f : A \to X$ is surjective (i.e., (-1)-connected), then X is small.

A generalization. (f n-connected means $\|\mathsf{fib}_f(x)\|_n \simeq 1$ for all x.)

Theorem 1 (C). Let $n \geq -2$. If A is small, X is $(n+2)$-locally small and $f : A \to X$ is n-connected, then X is small.

Ex. If G and H are groups, G is small and $f : BG \to_{\bullet} BH$ is 0-connected, then BH is small. (It is automatically 2-locally small.)

Restricting localizations

Assume $\mathcal{U} : \mathcal{U}'$, cumulative.

Theorem 2 (C). Let $n \geq -2$, and let L be a reflective subuniverse of $\mathcal{U}'$. If $\eta : X \to LX$ is n-connected and LX is $(n+2)$-locally small, for all $X : \mathcal{U}$, then the L-local types in $\mathcal{U}$ are reflective.

Proof sketch: By Theorem 1, LX is small for each $X : \mathcal{U}$.

For $X : \mathcal{U}$, let $\tilde{L}X : \mathcal{U}$ be equivalent to LX.

Define $\tilde{\eta}$ as the composite

$$X \xrightarrow{\eta} LX \simeq \tilde{L}X.$$

It is straightforward to check that this is a localization. □

Localization at a large family

Theorem 3 (C). Let f be any family of n-connected maps, for $n \geq -2$. The f-local $(n+1)$-types in $\mathcal{U}$ form a reflective subuniverse.

Proof sketch: Extend f to a family $\bar{f}$ by adding in the map $S^{n+2} \to 1$. The $\bar{f}$-local types are exactly the f-local $(n+1)$-types.

Let $\mathcal{U}'$ be a universe containing $\mathcal{U}$ and $\bar{f}$, and consider $L_{\bar{f}}$ on $\mathcal{U}'$.

By CORS, Theorem 3.12, each $\eta : X \to L_{\bar{f}}X$ is n-connected.

Every $(n+1)$-type is $(n+2)$-locally small, so Theorem 2 applies. □

Example. Applying this in Spaces to the family $B(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}) \to 1$ indexed by cardinals κ reproduces the CSS-localization.

Note. All results formalized using Coq-HoTT library.

Thanks for listening!