NON-ACCESSIBLE LOCALIZATIONS

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ABSTRACT. In a 2005 paper, Casacuberta, Scevenels and Smith construct a homotopy idempotent functor E on the category of simplicial sets with the property that whether it can be expressed as localization with respect to a map f is independent of the ZFC axioms. We show that this construction can be carried out in homotopy type theory. More precisely, we give a general method of associating to a suitable (possibly large) family of maps, a reflective subuniverse of any universe $\mathcal{U}$. When specialized to an appropriate family, this produces a localization which when interpreted in the ∞-topos of spaces agrees with the localization corresponding to E. Our approach generalizes the approach of [CSS] in two ways. First, by working in homotopy type theory, our construction can be interpreted in any ∞-topos. Second, while the local objects produced by [CSS] are always 1-types, our construction can produce n-types, for any n. This is new, even in the ∞-topos of spaces. In addition, by making use of universes, our proof is very direct.

Along the way, we prove many results about "small" types that are of independent interest. As an application, we give a new proof that separated localizations exist. We also give results that say when a localization with respect to a family of maps can be presented as localization with respect to a single map, and show that the simplicial model satisfies a strong form of the axiom of choice which implies that sets cover and that the law of excluded middle holds.

1. INTRODUCTION

In topology, the study of reflective subcategories (usually called "localizations") has a long history, beginning with work of Sullivan [Sul] and Bousfield [Bou] for topological spaces. This framework has played a fundamental organizing role in algebraic topology in the intervening decades, influencing and leading to the solution of many central conjectures in the field. More recently, the theory of reflective subcategories was adapted to the setting of ∞-categories by Lurie [Lur]. The key ingredient of a localization L is a function sending an object X to an object LX, with certain properties.

Independently, logicians and philosophers have considered the notion of modalities in logic, which allow one to qualify statements to express *possibility*, *necessity*, and other attributes such as *temporal* modalities. These modalities are expressed by applying a modal operator $\Diamond$ to a proposition P to produce a new proposition $\Diamond P$, with certain properties.

Homotopy type theory is a framework that can be used to study ∞-toposes. Thanks to recent work [BB, Boe, KL, LS, Shu], we know that anything proved in homotopy type theory is true in any ∞-topos[1]. The types in homotopy type theory are interpreted as *objects* of an ∞-topos, and are also used to encode *propositions* that one may wish to prove. It turns out that the notion of a *reflective subuniverse* in homotopy type theory simultaneously encodes the idea of a localization of an ∞-topos[2] and common modalities studied in logic [Cor].

[1]The initiality and semantics of higher inductive types still need to be fully worked out.

[2]More precisely, a reflective subuniverse in homotopy type theory corresponds to a family of reflective subcategories of each slice category, compatible with pullback (see [RSS, Appendix A] and [Ver]).

The foundations for the theory of reflective subuniverses in homotopy type theory were developed in [RSS]. We briefly describe the background from there needed to explain our results.

Background. All of our results are stated and proved within the framework of homotopy type theory. We follow the conventions and notation of [Uni]. Most of the results have been formalized using the Coq HoTT library [BGL+, HoTT] and are available at [Chr]; see Section 7 for details.

Let $\mathcal{U}$ be a univalent universe. Recall from [RSS, Uni] that a **reflective subuniverse** L of $\mathcal{U}$ consists of a subuniverse $\text{is-local}_L : \mathcal{U} \to \text{Prop}_{\mathcal{U}}$, a function $L : \mathcal{U} \to \mathcal{U}$, and a **localization** $\eta_X : X \to LX$ for each $X : \mathcal{U}$. Being a localization means that LX is L-local and η_X is initial among maps whose codomain is local.

Consider a family $f : \prod_{i:I}(A_i \to B_i)$ of maps. A type X is $\boldsymbol{f}$**-local** if precomposition with f_i

$$X^{B_i} \to X^{A_i}$$

is an equivalence for each $i : I$. A key result in the subject is [RSS, Theorem 2.18], which states that if the family f is in $\mathcal{U}$ in the sense that $I : \mathcal{U}$ and all $A_i, B_i : \mathcal{U}$, then the subuniverse of f-local types in $\mathcal{U}$ is reflective. A reflective subuniverse that is presented in this way is called **accessible**. (In classical topology, [Far] provides the foundations for such accessible localizations.)

Motivation. Given a reflective subuniverse L of $\mathcal{U}$, we can define a new subuniverse L' of $\mathcal{U}$ consisting of the types whose identity types are L-local. In [CORS], it is shown that L' is again reflective. In the proof, care was taken to avoid the assumption that L is accessible.

This raises the question of whether there are interesting non-accessible reflective subuniverses. In topology, work of [CSS] (building on [CGT, CRS]) shows that there exists a reflective subcategory of the ∞-topos of spaces whose accessibility is independent of ZFC. In their example, every local space is a 1-type. Their work is particularly interesting because it reveals a close connection to large cardinal axioms. They show that Vopěnka's principle implies that all localizations are accessible, which in turn implies the existence of measurable cardinals. The only example I am aware of in type theory is the double-negation modality, which is a reflection onto the subuniverse of stable propositions (those P such that $P \to \neg\neg P$ is an equivalence). However, this is accessible if one assumes propositional resizing, which holds in models associated to ∞-toposes [Shu, Proposition 11.3].

Main result. The goal of the present work is to give a construction of a reflective subuniverse in homotopy type theory which when interpreted in spaces reproduces the example of [CSS] whose accessibility is independent of ZFC. Our result is in fact more general. First, it gives a result in any model of homotopy type theory (e.g., in any ∞-topos). Second, it also produces examples at higher truncation levels. Moreover, by working within homotopy type theory, we are able to give a simpler proof, not at all following the classical proof. In future work, we hope to give a proof within homotopy type theory that our theorem can produce a reflective subuniverse that cannot be proven to be accessible (or even merely accessible).

The main result is:

Theorem 3.3. *Let $n \geq -1$ and let $f : \prod_{i:I}(A_i \to B_i)$ be a family of $(n-1)$-connected maps. Then the subuniverse of n-truncated f-local types in $\mathcal{U}$ is reflective.*

The key point is that none of I, A_i or B_i are required to be in $\mathcal{U}$. They can be in any other universe. When $n = 1$, this theorem reproduces the classical result of [CSS]. We explain this in Section 6. When $n > 1$, this result appears to be new, even for the ∞-topos of spaces. (We recall the definition of connectedness in Definition 2.1.)

In order to prove the main theorem, we first prove results that give conditions implying that a type is equivalent to a type in $\mathcal{U}$. Say that a type X is **small** if it is equivalent to a type in $\mathcal{U}$.

More precisely, X is **small** if $\sum_{X':\mathcal{U}} X' \simeq X$. Note that this is a mere proposition, by univalence. Define X to be **0-locally small** if it is small. Define X to be **$(n+1)$-locally small** if for all $x, x' : X$, $x = x'$ is n-locally small. These are also mere propositions.

Using his join construction [Rij], Rijke showed that if A is small, X is 1-locally small and $A \to X$ is (-1)-connected (surjective on π_0), then X is small. Rijke's result is the $n = 0$ case of the following result:

Proposition 2.2. *Let $n \geq -1$. If $f : A \to X$ is $(n-1)$-connected, A is small and X is $(n+1)$-locally small, then X is small.*

To prove the main result, we consider the extended family $\bar{f}$, indexed by $I + 1$, which also includes the map $S^{n+1} \to 1$. The $\bar{f}$-local types are exactly the n-truncated types which are f-local. We show that localization with respect to $\bar{f}$ on a universe $\mathcal{U}'$ that contains $\mathcal{U}$, I and all of the types A_i and B_i restricts to a localization on $\mathcal{U}$, using the proposition and [CORS, Theorem 3.12]. In addition, in order to adjust the predicate $\mathsf{is\text{-}local}_{\bar{f}}$ to land in $\mathcal{U}$, we need to use **propositional resizing**, the axiom that every mere proposition is small.

Organization. In Section 2, we prove a number of results about small and locally small types. These include the proposition above, as well as other results that are not needed for the main theorem, but may be of use in other situations. In Section 3, we prove the main theorem giving the existence of localizations determined by a large family of maps. We also show that the extended family $\bar{f}$ generates an orthogonal factorization system on $\mathcal{U}$. The short Section 4 uses some of the earlier results to give an alternative proof of the existence of L' localizations. Then in Section 5, we show that under certain assumptions, an accessible localization can be presented as a localization with respect to a single map. We then show that a strong form of the axiom of choice holds in the simplicial model, which implies the needed assumptions. In Section 6, we describe the relationship between our results and those of [CSS]. In Section 7, we briefly describe the formalization of these results, which is available at [Chr].

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2. Smallness

In this section, we prove a number of results about small types. The only results that will be used in the rest of the paper are Proposition 2.2 and Lemmas 2.3 and 2.4.

We defined "small" and "n-locally small" in the introduction. Note that these collections are Σ-closed: if Y is n-locally small and $A : Y \to \mathcal{U}'$ is a family of n-locally small types in some universe $\mathcal{U}'$, then $\sum_{y:Y} A(y)$ is n-locally small. These collections are also clearly closed under equivalence.

We will make frequent use of the concept of connectedness.

Definition 2.1. Let $k \geq -2$. A type A is **k-connected** if its k-truncation $\|A\|_k$ is contractible. A map $f : A \to B$ is **k-connected** if for every $b : B$, the fibre $\mathsf{fib}_f(b) :\equiv \sum_{a:A} (f(a) = b)$ is k-connected.

For example, the (-1)-connected maps are precisely the surjections, i.e., the maps that are surjective on π_0.

Using his join construction [Rij], Rijke showed that if A is small, X is 1-locally small and $A \to X$ is (-1)-connected, then X is small. We generalize this.

Proposition 2.2. *Let $n \geq -1$. If $f : A \to X$ is $(n-1)$-connected, A is small and X is $(n+1)$-locally small, then X is small.*

Proof. When $n = -1$, this is vacuous, since 0-locally small means the same as small.

Now let $n \equiv m + 1$ for $m \geq -1$. Assume $f : A \to X$ is $(n-1)$-connected, A is small and X is $(n+1)$-locally small. We'll show below that X is 1-locally small. Then since $(n-1)$-connected implies (-1)-connected, the inductive step will follow from the join construction.

Let $x, x' : X$. We need to show that $x = x'$ is small. Since being small is a mere proposition and f is surjective, we can assume that $x \equiv f(a)$ and $x' \equiv f(a')$ for some $a, a' : A$. We'll show that $f(a) = f(a')$ is small. We have $\mathsf{ap} f : a = a' \to f(a) = f(a')$. Note that $a = a'$ is small and this map is $(n-2)$-connected. By assumption, $f(a) = f(a')$ is n-locally small. So, by the induction hypothesis, $f(a) = f(a')$ is small.   □

Next we give some results that will be used to prove a dual version of Proposition 2.2.

Lemma 2.3. *Let $n \geq -1$. If X is n-truncated, then X is $(n+1)$-locally small.*

Proof. We prove this by induction on n. When $n = -1$, this is a restatement of propositional resizing, which we are assuming holds.

Now let $n \equiv m + 1$ with $m \geq -1$, and assume that X is $(m+1)$-truncated. This means that for each $x, y : X$, the identity type $x = y$ is m-truncated. By the induction hypothesis applied to $x = y$, we see that each $x = y$ is m-locally small. That is, X is $(m+1)$-locally small, as required.   □

Lemma 2.4. *Let $n \geq -1$. If $f : X \to Y$ is n-truncated and Y is $(n+1)$-locally small, then X is $(n+1)$-locally small.*

Recall that a map is said to be n-truncated if all of its fibres are n-truncated. For example, a (-1)-truncated map is exactly an embedding, so the $n = -1$ case of the lemma says that every subtype of a small type is small.

Proof. Note that X is equivalent to $\sum_{y:Y} \mathsf{fib}_f(y)$. The fibres are n-truncated by assumption, so by Lemma 2.3 they are $(n+1)$-locally small. The claim follows from the fact that the $(n+1)$-locally small types are Σ-closed.   □

Lemma 2.5. *Let $n \geq -1$. If $f : X \to Y$ is n-truncated, Y is $(n+1)$-locally small and X is n-connected, then X is small.*

Proof. By Lemma 2.4, X is $(n+1)$-locally small. Since $\|X\|_n \simeq 1$ and $n \geq -1$ we have that $\|X\|_{-1} \simeq 1$. Since we are proving a mere proposition, we can assume that we have $x : X$. Consider the map $1 \to X$ that selects the point x. Since X is n-connected, this map is $(n-1)$-connected [Uni, Lemma 7.5.11]. So by Proposition 2.2, X is small.   □

This allows us to characterize small types using truncations.

Theorem 2.6. *Let $n \geq -1$ and let X be a type. Then X is small if and only if X is $(n+1)$-locally small and $\|X\|_n$ is small.*

Proof. If X is small, then clearly it is $(n+1)$-locally small and its n-truncation is small.

To prove the converse, consider the truncation map $|-| : X \to \|X\|_n$ and note that

$$X \simeq \sum_{y:\|X\|_n} \sum_{x:X} (|x| = y).$$

Thus it suffices to show that the fibre $\sum_{x:X} (|x| = y)$ is small for each $y : \|X\|_n$. The first projection map $\sum_{x:X} (|x| = y) \to X$ is $(n-1)$-truncated, since the types $|x| = y$ are $(n-1)$-truncated. In particular, this map is n-truncated. Moreover, $\sum_{x:X} (|x| = y)$ is n-connected [Uni, Corollary 7.5.8]. Therefore, by Lemma 2.5, $\sum_{x:X} (|x| = y)$ is small, as required.   □

We can now give our dual version of Proposition 2.2, which generalizes Lemma 2.5.

Corollary 2.7. *Let $n \geq -1$. If $f : X \to Y$ is n-truncated, Y is $(n+1)$-locally small and $\|X\|_n$ is small, then X is small.*

Proof. By Lemma 2.4, X is $(n+1)$-locally small. So by Theorem 2.6, X is small. $\square$

The next result can be used to replace the condition that $\|X\|_n$ be small in Theorem 2.6 and Corollary 2.7 with a condition on homotopy groups.

Proposition 2.8. *Let $n \geq 0$ and let X be an n-truncated type. Then X is small if and only if $\pi_0(X)$ is small and for every $i : \mathbb{N}$ and $x : X$, $\pi_i(X, x)$ is small.*

The assumption on $\pi_i(X, x)$ is redundant when $i > n$, since X is n-truncated and contractible types are always small. Note that we separately assume that $\pi_0(X)$ is small, since X might not have any global elements.

Proof. The "only if" direction is clear.

We prove the converse by induction on n. When $n = 0$, X is equivalent to $\pi_0(X)$, which is assumed to be small.

Now assume that $n \equiv m + 1$ for $m : \mathbb{N}$. As in Theorem 2.6, consider the truncation map $|-| : X \to \|X\|_m$ and note that

$$X \simeq \sum_{y:\|X\|_m} \sum_{x:X} (|x| = y).$$

By the inductive hypothesis, $\|X\|_m$ is small. Again, we take $y \equiv |x'|$ for some $x' : X$, and need to show that the fibre $F :\equiv \sum_{x:X} (|x| = |x'|)$ is small. This type is pointed, m-connected and $(m+1)$-truncated, so it is an Eilenberg–Mac Lane space for some group G, by [BDR, Theorem 5.1]. From the long exact sequence of the fibre sequence, we see that $G \simeq \pi_{m+1}(X, x')$, which we have assumed is small. The construction of Eilenberg–Mac Lane spaces [LF] then implies that the fibre F is small, as required. $\square$

Example 2.9. Suppose that $f : BH \to_\bullet BG$ is a 0-truncated map. This means that its fibres are sets, or equivalently that the corresponding map $H \to G$ is a subgroup inclusion. If BG is small, then Lemma 2.5 implies that BH is small, since it is 0-connected.

Example 2.10. Suppose that $f : BG \to_\bullet BH$ is a 0-connected map, which means that it represents a surjection of groups [Uni, Corollary 8.8.5]. Suppose that BG is small. Since BH is a 1-type, it is 2-locally small. So by Proposition 2.2, BH is small.

3. Non-accessible localizations

This section contains the main results of the paper.

We will make use of the concept of an orthogonal factorization system. The following definition is equivalent to the definition in [RSS].

Definition 3.1. An **orthogonal factorization system on $\mathcal{U}$** is a pair of predicates $\mathcal{L}, \mathcal{R} : \prod_{X,Y:\mathcal{U}} (X \to Y) \to \mathsf{Prop}_{\mathcal{U}}$ such that $\mathcal{L} = {}^\perp\mathcal{R}$, $\mathcal{R} = \mathcal{L}^\perp$, and every map g in $\mathcal{U}$ can be factored as $g = k \circ h$, with h in $\mathcal{L}$ and k in $\mathcal{R}$. Here ${}^\perp\mathcal{R}$ denotes the class of maps with the unique left lifting property with respect to $\mathcal{R}$, and $\mathcal{L}^\perp$ denotes the class of maps with the unique right lifting property with respect to $\mathcal{L}$.

A key family of examples has as left class the n-connected maps and as right class the n-truncated maps, for each $n \geq -2$. (See [Uni, Section 7.6] or [RSS, Theorem 1.34].)

We begin with an easy result that we will use to prove the main theorem.

Proposition 3.2. *Let $\mathcal{U}$ be a universe contained in a universe $\mathcal{U}'$. Let L be a reflective subuniverse of $\mathcal{U}'$ such that for every $X : \mathcal{U}$, LX is small. Then the subuniverse of L-local types in $\mathcal{U}$ is reflective.*

Proof. By propositional resizing, we can replace the mere proposition $\mathsf{is\text{-}local}_L : \mathcal{U}' \to \mathsf{Prop}_{\mathcal{U}'}$ defining the subuniverse L with an equivalent one landing in $\mathsf{Prop}_{\mathcal{U}}$. Write $\mathsf{is\text{-}local}_{\tilde{L}} : \mathcal{U} \to \mathsf{Prop}_{\mathcal{U}}$ for the restriction of the new predicate to $\mathcal{U}$, giving us a subuniverse $\tilde{L}$ of $\mathcal{U}$.

We are given $\mathsf{is} : \prod_{X:\mathcal{U}} \sum_{X':\mathcal{U}} (X' \simeq LX)$. For $X : \mathcal{U}$, define $\tilde{L}X$ to be $\mathsf{pr}_1(\mathsf{is}\, X)$, so that $\tilde{L}X : \mathcal{U}$ and $\tilde{L}X \simeq LX$. In particular, $\tilde{L}X$ is in the subuniverse $\tilde{L}$.

Define $\tilde{\eta} : X \to \tilde{L}X$ to be the composite $X \to LX \to \tilde{L}X$, where the first map is the localization in $\mathcal{U}'$ and the second map is the inverse of the given equivalence. It is straightforward to check that $\tilde{\eta}$ is a localization. □

Now we give the main theorem.

Theorem 3.3. *Let $n \geq -1$ and let $f : \prod_{i:I} (A_i \to B_i)$ be a family of $(n-1)$-connected maps, with no smallness hypotheses on I, A_i or B_i. Then the subuniverse of n-truncated f-local types in $\mathcal{U}$ is reflective.*

Proof. Recall that a type is n-truncated if and only if it is local with respect to the single map $S^{n+1} \to 1$. Motivated by this, consider the extended family $\bar{f}$, indexed by $I + 1$, which also includes the map $S^{n+1} \to 1$. Then a type is $\bar{f}$-local if and only if it is f-local and n-truncated.

Let $\mathcal{U}'$ be a universe containing $\mathcal{U}$, I and all of the types A_i and B_i. By [RSS, Theorem 2.18], the subuniverse of $\bar{f}$-local types in $\mathcal{U}'$ is reflective. By Proposition 3.2, it is enough to show that for $X : \mathcal{U}$, $L_{\bar{f}}X$ is small. By Lemma 2.3, $L_{\bar{f}}X$ is $(n+1)$-locally small. Now, since all of the maps in the family $\bar{f}$ are $(n-1)$-connected (including the additional map $S^{n+1} \to 1$), and the $(n-1)$-connected maps are the left class in an orthogonal factorization system, [CORS, Theorem 3.12] implies that the map $\eta : X \to L_{\bar{f}}X$ is $(n-1)$-connected. Therefore, by Proposition 2.2, $L_{\bar{f}}X$ is small. □

The case $n = 1$ recovers the classical result of [CSS, Section 6], whose local objects are all 1-types. See Section 6 for more details.

Remark 3.4. As mentioned in the proof of Theorem 3.3, for each X, the type $L_{\bar{f}}X$ is n-truncated and the map $\eta_{\bar{f}} : X \to L_{\bar{f}}X$ is $(n-1)$-connected. It follows that for each $x : X$, the induced map

$$\pi_i(X, x) \longrightarrow \pi_i(L_{\bar{f}}X, \eta_{\bar{f}}x)$$

is an equivalence for $i < n$ and surjective for $i = n$, while for $i > n$, the codomain is trivial. As a consequence, if X is an $(n-1)$-type, then $\eta_{\bar{f}} : X \to L_{\bar{f}}X$ is an equivalence, by the truncated Whitehead theorem. In other words, every $(n-1)$-type is $\bar{f}$-local. So the $\bar{f}$-local types sit between the $(n-1)$-types and the n-types.

Remark 3.5. For a type X, consider the composite $X \to \|X\|_n \to L_f\|X\|_n$ of the localization maps. Just as for $\eta_{\bar{f}}$, both of these maps are $(n-1)$-connected. In particular, $L_f\|X\|_n$ is n-truncated and is therefore $\bar{f}$-local. Each of the localization maps has the property that it is sent to an equivalence by $Z^{(-)}$ for any $\bar{f}$-local type Z. Indeed, the first is initial among maps into all n-truncated types, and the second is initial among maps into all f-local types. Therefore, this composite also presents the $\bar{f}$-localization of X. That is, there is an equivalence $e : L_{\bar{f}}X \to L_f\|X\|_n$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;|-|\;} & \|X\|_n \\
{\scriptstyle \eta_{\bar{f}}} \downarrow & & \downarrow {\scriptstyle \eta_f} \\
L_{\bar{f}}X & \xrightarrow[\;e\;]{\sim} & L_f\|X\|_n
\end{array}
$$

commute.

Under the same hypotheses as Theorem 3.3, we can get an orthogonal factorization system on $\mathcal{U}$. This possibility was suggested to me by Mathieu Anel.

Theorem 3.6. *Let $n \geq -1$ and let $f : \prod_{i:I}(A_i \to B_i)$ be a family of $(n-1)$-connected maps. Let $\mathcal{R}$ consist of those maps in $\mathcal{U}$ which are n-truncated and right orthogonal to f, and let $\mathcal{L}$ consist of those maps in $\mathcal{U}$ which are left orthogonal to $\mathcal{R}$. Then $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorization system on $\mathcal{U}$.*

Proof. The proof parallels the proof of Theorem 3.3. Recall that a map is n-truncated if and only if it is right orthogonal to the single map $S^{n+1} \to 1$. Extend f to a family $\bar{f}$ of maps indexed by $I + 1$, which also includes the map $S^{n+1} \to 1$. Then a map is right orthogonal to $\bar{f}$ if and only if it is n-truncated and right orthogonal to f.

Let $\mathcal{U}'$ be a universe containing $\mathcal{U}$, I and all of the types A_i and B_i in the family f. By [RSS, Theorem 2.41], there is an orthogonal factorization system $(\mathcal{L}', \mathcal{R}')$ on $\mathcal{U}'$, with $\mathcal{R}' = \bar{f}^{\perp}$ and $\mathcal{L}' = {}^{\perp}\mathcal{R}'$. Let $g : X \to Y$ be a map in $\mathcal{U}$, and consider an $(\mathcal{L}', \mathcal{R}')$-factorization

$$X \xrightarrow{l} W \xrightarrow{r} Y$$

of g. We will show below that W is small. Therefore, after composing with appropriate equivalences, we can assume that W is in $\mathcal{U}$. It follows that r is in $\mathcal{R}$, since $\mathcal{R}$ consists precisely of those maps in $\mathcal{R}'$ which are in $\mathcal{U}$. The map l is left orthogonal to $\mathcal{R}'$, so it is also left orthogonal to $\mathcal{R}$, and hence is in $\mathcal{L}$.

We now show that W is small. Since r is in $\mathcal{R}'$, it is n-truncated. Therefore, by Lemma 2.4, W is $(n+1)$-locally small. Since every map in $\bar{f}$ is $(n-1)$-connected, $\mathcal{R}'$ contains the $(n-1)$-truncated maps. Therefore, $\mathcal{L}'$ is contained in the $(n-1)$-connected maps, and so l is $(n-1)$-connected. Proposition 2.2 then implies that W is small. $\square$

Theorem 3.3 follows from Theorem 3.6 by considering the factorization of the map $X \to 1$ for each X in $\mathcal{U}$.

4. AN APPLICATION TO SEPARATED LOCALIZATIONS

In this short section, we give an alternate proof of a result from [CORS]. This is not needed in the remainder of the paper.

We first recall some terminology. For a reflective subuniverse L, a type X is **L-separated** if $x = y$ is L-local for every $x, y : X$.

Theorem 4.1 ([CORS, Theorem 2.25]). *For any reflective subuniverse L of $\mathcal{U}$, the subuniverse of L-separated types in $\mathcal{U}$ is again reflective.*

Proof. Consider the type of L-equivalences in $\mathcal{U}$, namely $I :\equiv \Sigma_{X:\mathcal{U}}\Sigma_{Y:\mathcal{U}}\Sigma_{g:X \to Y}\mathsf{IsEquiv}(Lg)$. Projection onto the third component g defines a family f of maps. By [CORS, Lemma 2.9], the maps $\eta : X \to LX$ are L-equivalences, and so the f-local types in $\mathcal{U}$ are exactly the L-local types.

Now consider the family Σf which sends $(X; Y; g; p)$ to the suspension $\Sigma g : \Sigma X \to \Sigma Y$ of g. By [CORS, Lemma 2.15], a type in $\mathcal{U}$ is (Σf)-local if and only if it is L-separated.

Let $\mathcal{U}'$ be a universe containing $\mathcal{U}$, and therefore the type I of L-equivalences. By [RSS, Theorem 2.18], the subuniverse of (Σf)-local types in $\mathcal{U}'$ is reflective. By Proposition 3.2, it suffices to show that for each $X : \mathcal{U}$, $L_{\Sigma f}X$ is small. Each Σg is (-1)-connected, so by [CORS, Theorem 3.12] the maps $\eta' : X \to L_{\Sigma f}X$ are (-1)-connected. Therefore, by Rijke's join construction (the $n = 0$ case of Proposition 2.2), it suffices to show that $L_{\Sigma f}X$ is 1-locally small when $X : \mathcal{U}$, i.e. that $x' = y'$ is small for $x', y' : L_{\Sigma f}X$. Since this is a proposition and η' is surjective, it is enough

to prove that $\eta'(x) = \eta'(y)$ is small for $x, y : X$. But by [CORS, Proposition 2.26] (which does not use Theorem 2.25), $(\eta'(x) = \eta'(y)) \simeq L(x = y)$, which is in $\mathcal{U}$. $\qquad\square$

This proof uses many of the same ingredients as the proof in [CORS], but is a bit more conceptual.

5. Replacing a family with a single map

In homotopy type theory, an accessible reflective subuniverse can be presented as a localization with respect to a family of maps indexed by a type. Here we investigate when the family can be replaced by a single map. We give some results internally in Section 5.1 and then in the simplicial model in Section 5.2.

5.1. **In homotopy type theory.** We begin with a reduction to a family indexed by a 0-type.

Lemma 5.1. *Let $f : \Pi_{i:I} A_i \to B_i$ be a family of maps indexed by a type I, and let $g : J \to I$ be an epimorphism. Then a type X is f-local if and only if it is $(f \circ g)$-local, where*

$$(f \circ g)(j) :\equiv f_{g(j)} : A_{g(j)} \to B_{g(j)}.$$

Proof. It is clear that if X is f-local, then it is $(f \circ g)$-local. (This direction does not use that g is surjective.)

Conversely, suppose that X is $(f \circ g)$-local. Let $i : I$. We would like to prove that $(f_i)^* : X^{B_i} \to X^{A_i}$ is an equivalence. This is a proposition, so we may assume we have $j : J$ with $g(j) = i$. Since $f_{g(j)}^*$ is an equivalence by assumption, we are done. $\qquad\square$

Definition 5.2. The axiom **sets cover** says that for every type X, there merely exists a 0-type S and an epimorphism $S \to X$. For a universe $\mathcal{U}$, we say that **sets cover in $\mathcal{U}$** if for every $X : \mathcal{U}$, there merely exists a 0-type $S : \mathcal{U}$ and an epimorphism $S \to X$.

We show in Section 5.2 that this holds in the simplicial model.

Proposition 5.3. *Assume that sets cover in $\mathcal{U}$. Then any accessible reflective subuniverse L of $\mathcal{U}$ can merely be presented as localization with respect to a family in $\mathcal{U}$ indexed by a set.*

Proof. Suppose L is presented as localization with respect to a family f indexed by a type $I : \mathcal{U}$. Since we are proving a proposition, we can choose an epimorphism $g : S \to I$ from a set S in $\mathcal{U}$. Then L can be presented as localization with respect to the family $f \circ g$ indexed by S, using Lemma 5.1. $\qquad\square$

Next we consider reducing from a set-indexed family to a single map. Given a family $f : \Pi_{i:I} A_i \to B_i$ of maps indexed by a type I, write $\mathsf{tot}(f) : (\Sigma_i A_i) \to (\Sigma_i B_i)$ for the induced map on Σ-types. While every f-local type X is also $\mathsf{tot}(f)$-local, the converse is only true under some assumptions.

Proposition 5.4. *Let $f : \Pi_{i:I} A_i \to B_i$ be a family of maps. Then every f-local type is $\mathsf{tot}(f)$-local. If I is a set with decidable equality and X is merely inhabited, then X is f-local if and only if X is $\mathsf{tot}(f)$-local.*

Proof. Suppose X if f-local. To show that X is $\mathsf{tot}(f)$-local, we must show that the natural map

$$\left(\left(\sum_i A_i\right) \to X\right) \longleftarrow \left(\left(\sum_i B_i\right) \to X\right)$$

is an equivalence. This map is equivalent to the natural map

$$\left(\prod_i X^{A_i}\right) \longleftarrow \left(\prod_i X^{B_i}\right), \tag{1}$$

which is an equivalence since each component is.

Now assume that I is a set with decidable equality and that X is merely inhabited and $\mathsf{tot}(f)$-local. Then (1) is an equivalence. Let $j : I$. We must show that the map

$$X^{A_j} \leftarrow X^{B_j} \tag{2}$$

is an equivalence. Since this is a proposition, we can assume that we have $x_0 : X$. We proceed by showing that (2) is a retract of (1). There are natural projection maps from (1) to (2) given by evaluation at j. Under our assumptions, these have sections. For example, we define $X^{A_j} \to \prod_i X^{A_i}$ by sending f to the function sending i to f when $i = j$ and to the constant map at x_0 otherwise. We define a similar map from X^{B_j}. These are sections of the evaluation maps and make the necessary square commute, showing that (2) is a retract of (1). $\qquad\square$

Proposition 5.5. *Let $f : \Pi_{i:I} A_i \to B_i$ be a family of maps indexed by a set I such that the empty type is f-local. Assuming the law of excluded middle for propositions (LEM), the f-local types and the $\mathsf{tot}(f)$-local types agree.*

Note that if A_i and B_i are merely inhabited for each i, then the empty type is f-local.

Proof. By Proposition 5.4, every f-local type is $\mathsf{tot}(f)$-local. We must prove the converse. Suppose that X is $\mathsf{tot}(f)$-local. By LEM applied to $\|X\|_{-1}$, we know that either X is merely inhabited or we have $\|X\|_{-1} \to \emptyset$. If X is merely inhabited, then Proposition 5.4 shows that X is f-local. (Since we are assuming LEM, I has decidable equality.)

On the other hand, if we have $\|X\|_{-1} \to \emptyset$, then we have $X \to \emptyset$ which implies that $X = \emptyset$. Therefore, X is f-local, as required. $\qquad\square$

Combining these propositions gives:

Theorem 5.6. *If sets cover in $\mathcal{U}$, the law of excluded middle holds and L is an accessible reflective subuniverse such that the empty type is L-local, then L can merely be presented as localization with respect to a single map.* $\qquad\square$

Remark 5.7. By Remark 3.4, any localization L produced by Theorem 3.3 with $n \geq 0$ has the property that $\emptyset$ (and all propositions) are L-local. So, under the assumptions of Theorem 5.6, if such a localization is accessible, then there merely exists a presentation using a single map.

5.2. **Properties of the simplicial model.** In this section, we show that the simplicial model [KL3] satisfies a strong form of the axiom of choice, which implies in particular that sets cover and that the law of excluded middle holds. These results are known to others. The reference [KL2] proves that the law of excluded middle holds in the simplicial model, but we know of no references for the other facts, so we include the proofs here to be self-contained.

Given a fibration $f : X \to \Gamma$ of simplicial sets and a vertex v of Γ, we write $v : 1 \to \Gamma$ for the vertex inclusion and v^*f for the fibre $f^{-1}(v)$ of f over v. We say that a fibration or fibrant object is k-connected or k-truncated if it satisfies the interpretation of the homotopy type theory definition, e.g. Definition 2.1.

Lemma 5.8. *A fibrant simplicial set X is (-1)-truncated (resp. (-1)-connected) if and only if it is empty or contractible (resp. non-empty).*

Proof. It is straightforward to check that in the empty context the definitions specialize to "empty or contractible" and "non-empty," respectively. $\qquad\square$

Lemma 5.9. *Let $f : X \to \Gamma$ be a fibration between fibrant simplicial sets. If f is (-1)-truncated (resp. (-1)-connected), then for each vertex v of Γ, the fibre v^*f is empty or contractible (resp. non-empty).*

Proof. This follows immediately from Lemma 5.8, since being (-1)-truncated (resp. (-1)-connected) is stable under pullback. □

Proposition 5.10. *Let $p : P \to \Gamma$ be a (-1)-truncated fibration between fibrant simplicial sets. Then p has a section if and only if the fibre v^*p is non-empty for each vertex v of Γ.*

Proof. By Lemma 5.9, the fibres v^*p of p are empty or contractible. Suppose that v^*p is non-empty for each vertex v. Then these fibres must be contractible, so p is a trivial fibration and therefore has a section. The converse is trivial. □

This lets us prove the converse of Lemma 5.9.

Proposition 5.11. *Let $f : X \to \Gamma$ be a fibration between fibrant simplicial sets. Then f is (-1)-truncated (resp. (-1)-connected) if and only if for each vertex v of Γ, the fibre v^*f is empty or contractible (resp. non-empty).*

Proof. One direction is given by Lemma 5.9. For the converse, suppose that each v^*f is non-empty. The condition that f be a (-1)-connected map says that a certain fibration $C \to \Gamma$ has a section, where C is the interpretation of $\Sigma_y\|\mathsf{fib}_f(y)\|_{-1}$. Since $C \to \Gamma$ is a family of mere propositions, the map $C \to \Gamma$ is (-1)-truncated. On the other hand, the assumption that each v^*f is non-empty tells us that the fibres of $C \to \Gamma$ are non-empty, since the interpretation of C commutes with pullback to each vertex. So it follows from Proposition 5.10 that $C \to \Gamma$ has a section.

A similar argument works for (-1)-truncated maps. □

We will now see how these propositions let us reduce many questions to the empty context.

For $-1 \leq n \leq \infty$, let AC_n be the statement that for every 0-type X in some context Γ, every family of merely inhabited n-types over X merely has a section. When $n = \infty$, there is no truncation condition on the family. This is $\mathrm{AC}_{n,-1}$ from [Uni, Exercise 7.8], but phrased in a way that avoids universes. The following result is mentioned in that exercise, without proof.

Theorem 5.12. *The simplicial model satisfies AC_∞. More precisely, for every 0-truncated fibration $p : X \to \Gamma$ and every (-1)-connected fibration $e : Y \to X$, the (-1)-truncation of the fibration $\Pi_p e$ in the slice over Γ has a section.*

Proof. Let p and e be as in the statement. We need to show that the (-1)-truncation of the fibration $\Pi_p e$ has a section. In other words, we need to show that $\Pi_p e$ is a (-1)-connected map. By Proposition 5.11, it is enough to show that for each vertex v of Γ, the preimage $v^*(\Pi_p e)$ is non-empty. Since Π commutes with pullbacks, this preimage is equivalent to $\Pi_{v^*p}v^*e$, which puts us in the empty context. Here v^*e denotes the functorial action of pullback, giving a (-1)-connected fibration over a homotopically discrete simplicial set v^*p. Using the external axiom of choice, choose one vertex of each component to obtain a trivial cofibration $S \to v^*p$ from a discrete simplicial set. Using choice again, we can lift this map as shown, since v^*e is (-1)-connected:

$$
\begin{array}{ccc}
S & \dashrightarrow & v^*(e \circ p) \\
\downarrow & & \downarrow{\scriptstyle v^*e} \\
v^*p & =\!=\!=\!= & v^*p.
\end{array}
$$

Finally, we use the lifting property of trivial cofibrations and fibrations to obtain the desired section of v^*e. □

As another example, we give an alternate proof of the following result.

Proposition 5.13 ([KL3])**.** *The law of excluded middle holds in the simplicial model. That is, for every* (-1)*-truncated fibration* $q : A \to \Gamma$ *of simplicial sets, the fibration* $p : P \to \Gamma$ *corresponding to* $\mathrm{LEM}(A) :\equiv A + \neg A$ *has a section.*

Proof. Note that p is (-1)-truncated, since $\mathrm{LEM}(A)$ is a mere proposition when A is. So, by Proposition 5.10, it is enough to show that v^*p is non-empty for each vertex v of Γ. Since LEM commutes with pullback, v^*p is equivalent to $\mathrm{LEM}(v^*q)$. In other words, we have reduced the problem to the empty context. But for any fibrant simplicial set X, $X + \neg X$ has a global element, since X is either non-empty or empty. $\qquad\square$

The previous result also follows from Diaconescu's Theorem [Uni, Theorem 10.1.14] and Theorem 5.12, since AC_∞ clearly implies AC_0.

Corollary 5.14. *For any universe* $\mathcal{U}$ *in the simplicial model, sets cover in* $\mathcal{U}$*.*

It follows that Remark 5.7 applies in the simplicial model.

Proof. By [Uni, Exercise 7.9], which has been formalized by Jarl Flaten in [HoTT], this follows from Theorem 5.12. The set that covers X can be taken to be $\pi_0(X)$, so it lives in the same universe as X. $\qquad\square$

Alternatively, one can use the methods above to show that it suffices to check this in the empty context. Then, using choice, one can also find a (-1)-connected map $\pi_0(X) \to X$.

The theme of this section is that many questions can be reduced to the empty context. This is fundamentally because of Proposition 5.11, which says that (-1)-connected maps are determined by their pullbacks to the terminal object, and corresponds to the model being "well-pointed" in some sense. Note that not every question can be checked in the empty context. For example, for any fibrant simplicial set X, $\mathrm{LEM}(X)$ holds. But this does not hold for families.

6. The relationship to past work

6.1. **The local objects.** The paper [CSS] works in the category of simplicial sets. The local objects in [CSS, Theorem 6.4] are the 1-types X such that for every (small) infinite cardinal κ and every $x \in X$,
$$\mathrm{Hom}(\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}, \pi_1(X, x)) = 1.$$
Here, $\mathbb{Z}^\kappa$ is the abelian group of all functions $f : \kappa \to \mathbb{Z}$, and $\mathbb{Z}^{<\kappa}$ is the subgroup consisting of those functions f whose support has cardinality smaller than κ. In homotopy type theory, we can choose a family of groups that gives the same condition when interpreted in simplicial sets, where the law of excluded middle holds for propositions (see [KL2] and Proposition 5.13). For example, one could index the family by 0-types K in $\mathcal{U}$ which are "infinite" in some sense, e.g., in that they merely have a self-map that is injective but not surjective. For each such K, one considers the group $\mathbb{Z}^K/\mathbb{Z}^{<K}$, where $\mathbb{Z}^K :\equiv (K \to \mathbb{Z})$ and $\mathbb{Z}^{<K}$ could be defined to be the subgroup generated by those f satisfying
$$\neg\left(\sum_{g:\ \mathrm{supp}(f)\to K} \mathsf{is}\text{-}(-1)\text{-}\mathsf{connected}(g)\right).$$
Variations are possible and we make no claim that these definitions are the most practical for future work. We only claim that in the simplicial model, these produce the same family as used in [CSS] (with repetition, which is harmless).

More generally, one can consider a class S of epimorphisms of small groups and define the S-local objects to be the 1-types X such that for every $f : A \to C$ in S and every $x \in X$, the natural map
$$\mathrm{Hom}(A, \pi_1(X, x)) \leftarrow \mathrm{Hom}(C, \pi_1(X, x))$$

is a bijection. (Taking the class of maps $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \to 1$ reproduces the previous case.) We can characterize these local objects in the following way, using that for a 1-type X, $\mathrm{Hom}(A, \pi_1(X, x)) \simeq (BA \to_\bullet (X, x))$:

$$\{S\text{-local objects}\} = \{1\text{-types } X \mid \prod_{f:A\to C} \prod_{x:X} (BA \to_* (X, x)) \leftarrow (BC \to_* (X, x)) \text{ is an equiv.}\}$$

$$= \{1\text{-types } X \mid \prod_{f:A\to C} (BA \to X) \leftarrow (BC \to X) \text{ is an equiv.}\}$$

$$= \{1\text{-types } X \mid X \text{ is local w.r.t. } \{Bf : BA \to BC \mid f \in S\}\}.$$

Two of the products are over $f \in S$, and the first two maps are induced by $Bf : BA \to_\bullet BC$. The second equality is [CORS, Lemma 3.3]. (We are using type-theoretic notation here, but this can all be interpreted classically.) When f is a surjection of groups, Bf is a 0-connected map, so Theorem 3.3 gives a localization onto the types described in the last displayed line. (Note that the internal and external homs agree for simplicial sets, so there is no need to distinguish between "internally" and "externally" local objects.)

We next discuss the interpretation of this in more detail.

6.2. **The localization.** We will compare the results of this paper with [CSS] by comparing what they give in the ∞-topos of spaces, i.e., in the simplicial nerve $N(\mathrm{Kan})$ of the simplicial category Kan of Kan complexes. (This is also called the homotopy coherent nerve.)

The paper [CSS] was written before the theory of ∞-categories was well-established. For a class S of surjections of small groups, they construct a **homotopy idempotent functor** E whose local objects are the S-local spaces. More precisely, E is a (strict) functor sSet $\to$ sSet which sends weak equivalences to weak equivalences and is equipped with a (strict) natural transformation $\eta_X : X \to EX$ such that, for each X, η_{EX} and $E\eta_X$ are weak equivalences $EX \to EEX$.[3] (They note that it follows that η_{EX} and $E\eta_X$ are homotopic, and that one can assume without loss of generality that EX is a Kan complex for every simplicial set X.)

By [RSS', Corollary 6.5], one can replace E by a weakly equivalent *simplicial* functor sSet $\to$ sSet. One can then apply the simplicial nerve functor N to obtain an (∞-)functor E' on the ∞-topos $N(\mathrm{Kan})$ of spaces, which gives a localization in Lurie's sense (see [Lur, Proposition 5.2.7.4]). The essential image of E' consists of the S-local Kan complexes.

Now we consider how Theorem 3.3 gives rise to a localization in Lurie's sense. There are several subtleties to deal with, and we only give an outline. Let $\beta < \alpha$ be (strongly) inaccessible cardinals. By [KL3], there is a model of homotopy type theory based on the category of α-small simplicial sets (those X such that each X_n has cardinality less than α) and which contains an internal universe U_β corresponding to the β-small simplicial sets. We regard the β-small simplicial sets as the usual "small" simplicial sets, and the larger simplicial sets as a technical tool. ([CSS] also makes use of such larger simplicial sets.) We take U_β as our interpretation of the universe $\mathcal{U}$ appearing in Theorem 3.3.

Theorem 3.3 produces a reflective subuniverse of $\mathcal{U}$. While we expect that one could interpret this directly as a localization of the ∞-topos of β-small Kan complexes, this has not been written down explicitly. The Appendix of [RSS] instead gives the interpretation of a "judgmental reflective subuniverse" that acts on *all* types, not just those in some universe. The reflective subuniverse $L_{\bar{f}}$ that appears in the proof of Theorem 3.3 is polymorphic in this sense, as it is accessible. Therefore, [RSS, Theorem A.12] applies and tells us that we get a reflective subfibration of the ∞-topos of α-small Kan complexes, which in particular gives a localization in Lurie's sense. The

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[3]It is because of this strictness that we compare our results in the ∞-topos of spaces. It is not clear whether a general localization of the ∞-topos of spaces can be "strictified" into a homotopy coherent functor. Most results along these lines deal with accessible localizations.

localization produced by Theorem 3.3 is then interpreted as the restriction of this localization to the sub-∞-topos of Kan complexes which are β-small (up to equivalence).

As a localization is determined by its essential image, the discussion of the previous section shows that this localization agrees with the localization coming from the work of [CSS].

6.3. **Accessibility.** By Theorem 5.6, Remark 5.7 and the results of Section 5.2, if the localization produced by Theorem 3.3 is accessible in homotopy type theory, then it can merely be presented as localization with respect to a single map. In particular, it would follow that the local spaces described in Section 6.1 are f-local for a single map f between simplicial sets. This is what [CSS] show is not provable from ZFC (assuming that ZFC is consistent), and so it follows that the localization in homotopy type theory also cannot be proven to be accessible.

Note also that [RSS, Theorem A.20] shows that if L is an accessible reflective subuniverse in homotopy type theory, then the corresponding reflective subcategory in an ∞-topos model is accessible. This does not itself guarantee that it can be presented using a single map, but the arguments of Section 5 can be carried out directly in the simplicial model to show that this is the case.

7. Formalization

The main results of this paper have been formalized using the Coq HoTT library [HoTT] and are available at [Chr]. Specifically, at the time of writing, the following results have been formalized: Proposition 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5, Theorem 2.6, Corollary 2.7, Proposition 3.2, and Theorem 3.3. The `README.md` file at [Chr] gives the mapping between the results here and the formalized results and will be updated as more results are formalized. After further polishing, the results will be pushed to the main library.

The formalization of the above results takes approximately 500 lines. In order to do these formalizations, a number of other known results needed to be formalized. For example, we prove a number of results about connected maps, enough to show that if f is a surjection between groups, then Bf is a 0-connected map. (See Example 2.10 and [Uni, Corollary 8.8.5].) We also prove a special case of [CORS, Theorem 3.12], namely that if O is a modality and L_f is the localization with respect to a family with each f_i O-connected, then each $\eta : X \to L_f X$ is O-connected. This was challenging because [RSS, Lemma 1.44] has not been formalized, so we had to also prove various results about orthogonal factorization systems. The results described in this paragraph required about 600 additional lines.

In addition, the $n = 0$ case of Proposition 2.2, which was proved informally by Rijke [Rij], is assumed as an axiom in the formalization. It has been formalized in Arend by Valery Isaev [Isa] (see the file `Homotopy/Image.ard`).

References

[BGL+] A. Bauer, J. Gross, P. LeFanu Lumsdaine, M. Shulman, M. Sozeau, and B. Spitters. The HoTT Library: A Formalization of Homotopy Type Theory in Coq. In: Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs, CPP 2017, Paris, France, ACM, 2017, 164–172.

[BB] M. de Boer and G. Brunerie. Agda formalization of the initiality conjecture, 2020. https://github.com/guillaumebrunerie/initiality

[BDR] U. Buchholtz, F. van Doorn, and E. Rijke. Higher Groups in Homotopy Type Theory. In: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '18, Oxford, ACM, 2018, 205–214.

[Boe] M. de Boer. A proof and formalization of the initiality conjecture of dependent type theory. Licentiate thesis, Department of Mathematics, Stockholm University, 2020. http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-181640

[Bou] A.K. Bousfield. The localization of spaces with respect to homology. Topology **14** (1975), 133–150.

[CGT]   C. Casacuberta, M. Golasiński, and A. Tonks. Homotopy localization of groupoids. Forum Math. **18** (2006), 967–982.

[Chr]   J.D. Christensen. Formalization of non-accessible localizations in Coq, 2021. https://github.com/jdchristensen/NonAccessible

[Cor]   D. Corfield. Modal homotopy type theory: The prospect of a new logic for philosophy. Oxford University Press, 2020, 192 pp.

[CORS] J.D. Christensen, M.P. Opie, E. Rijke, and L. Scoccola. Localization in homotopy type theory. High. Struct. **4**(1) (2020), 1–32.

[CRS]   C. Casacuberta, J.L. Rodríguez, and D. Scevenels. Singly generated radicals associated with varieties of groups. In: Groups St. Andrews 1997 in Bath, I, 202–210, London Math. Soc. Lecture Note Ser., 260, Cambridge Univ. Press, Cambridge, 1999.

[CSS]   C. Casacuberta, D. Scevenels, and J.H. Smith. Implications of large-cardinal principles in homotopical localization. Adv. Math. **197**(1) (2005), 120–139.

[Far]   E.D. Farjoun. Cellular spaces, null spaces and homotopy localization. Vol. 1622. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996, xiv+199 pp.

[HoTT] The Coq HoTT Library. https://github.com/HoTT/HoTT

[Isa]   V. Isaev. Arend Standard Library. https://github.com/JetBrains/arend-lib

[KL]   K. Kapulkin and P. LeFanu Lumsdaine. The homotopy theory of type theories. Adv. Math. **338** (2018), 1–38.

[KL2]   K. Kapulkin and P. LeFanu Lumsdaine. The law of excluded middle in the simplicial model of type theory. Theory Appl. Categ. **35**(40) (2020), 1546–1548.

[KL3]   K. Kapulkin and P. LeFanu Lumsdaine. The Simplicial Model of Univalent Foundations (after Voevodsky). J. Eur. Math. Soc. **23**(6) (2021), 2071–2126.

[LF]   D.R. Licata and E. Finster. Eilenberg-MacLane spaces in homotopy type theory. In: Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), ACM, New York, 2014, Article No. 66.

[LS]   P. LeFanu Lumsdaine and M. Shulman. Semantics of higher inductive types. Math. Proc. Cambridge Philos. Soc. **169**(1) (2020), 159–208.

[Lur]   J. Lurie. Higher topos theory. Annals of Mathematics Studies 170. Princeton University Press, 2009.

[Rij]   E. Rijke. The join construction, 2017. Arxiv: 1701.07538v1

[RSS]   E. Rijke, M. Shulman, and B. Spitters. Modalities in homotopy type theory. In: Log. Methods Comput. Sci. **16**(1) (2020), Paper No. 2, 79 pp.

[RSS'] C. Rezk, S. Schwede, B. Shipley. Simplicial structures on model categories and functors. Amer. J. Math. **123** (2001), 551–575.

[Shu]   M. Shulman. All $(\infty, 1)$-toposes have strict univalent universes, 2019. Arxiv: 1904.07004v2

[Sul]   D.P. Sullivan. Geometric topology: localization, periodicity and Galois symmetry. The 1970 MIT notes. Edited and with a preface by Andrew Ranicki. K-Monographs in Mathematics **8**. Springer, Dordrecht, 2005.

[Uni]   The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, Princeton, NJ, 2013. http://homotopytypetheory.org/book

[Ver]   M. Vergura. Localization theory in an ∞-topos. 2019. Arxiv: 1907.03836v1

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