FAILURE OF BROWN REPRESENTABILITY IN DERIVED CATEGORIES

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dedicated to H. Lenzing on the occasion of his sixtieth birthday

ABSTRACT. Let $\mathcal{T}$ be a triangulated category with coproducts, $\mathcal{T}^c \subset \mathcal{T}$ the full subcategory of compact objects in $\mathcal{T}$. If $\mathcal{T}$ is the homotopy category of spectra, Adams proved the following in [1]: All homological functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ are the restrictions of representable functors on $\mathcal{T}$, and all natural transformations are the restrictions of morphisms in $\mathcal{T}$.

It has been something of a mystery, to what extent this generalises to other triangulated categories. In [36], it was proved that Adams' theorem remains true as long as $\mathcal{T}^c$ is countable, but can fail in general. The failure exhibited was that there can be natural transformations not arising from maps in $\mathcal{T}$.

A puzzling open problem remained: Is every homological functor the restriction of a representable functor on $\mathcal{T}$? In a recent paper, Beligiannis [5] made some progress. But in this article, we settle the problem. The answer is no. There are examples of derived categories $\mathcal{T} = D(R)$ of rings, and homological functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ which are not restrictions of representables.

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INTRODUCTION

The introduction is written for the reader who knows about derived categories, but is not necessarily familiar with previous articles by the authors and their close friends. We begin with a sketch of the work done in the last 10 years, generalising results from homotopy theory to derived categories. The experts may want to skip this, and go directly to Notation 0.4 on page 4. After the very general survey, will come a much more focused one. We will give, in some detail, the history of the results on generalising

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the theorem of Brown and Adams to derived categories. Then we will explain the two open problems, which we settle in this article. Finally, we will give the nature of our counterexamples.

Let $\mathcal{T}$ be a triangulated category. The representable functors $\mathcal{T}(-,X)$ are all homological; that is, they take triangles to long exact sequences. Given a triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$, we can restrict a representable functor on $\mathcal{T}$ to a functor on $\mathcal{S}$. We denote the restriction by $\mathcal{T}(-,X)\big|_{\mathcal{S}}$. All such functors are clearly homological.

The most interesting version of this, is where $\mathcal{T}$ is a triangulated category with coproducts, and $\mathcal{S}$ is the full subcategory $\mathcal{T}^c$ of all compact objects in $\mathcal{T}$.

Definition 0.1. *An object $c \in \mathcal{T}$ is called* compact, *if the functor $\mathcal{T}(c,-)$ commutes with coproducts.*

We remind the reader that for $\mathcal{T}$ the homotopy category of spectra, $\mathcal{T}^c \subset \mathcal{T}$ is the subcategory of finite spectra. For $\mathcal{T} = D(R)$, the unbounded derived category of right R–modules, $\mathcal{T}^c$ turns out to be the subcategory of perfect complexes, that is, complexes isomorphic to finite complexes of finitely generated projective R–modules. For a more detailed discussion of examples, where $\mathcal{T}$ is the unbounded derived category of coherent sheaves on a scheme, see Sections 1 and 2 in [35].

Since the functor $\mathcal{T}(-,X)\big|_{\mathcal{T}^c}$ plays a major role in what follows, we adopt a shorthand for it. We will write

$$\mathbf{y}X = \mathcal{T}(-,X)\big|_{\mathcal{T}^c}.$$

The subject we will be studying began with a theorem of Adams [1].

Theorem 0.2. (Adams, 1971) *Let $\mathcal{T}$ be the homotopy category of spectra, and $\mathcal{T}^c$ the subcategory of finite spectra. Then any homological functor $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ is isomorphic to $\mathbf{y}X$, for some object $X \in \mathcal{T}$. Furthermore, any natural transformation of functors*

$$\mathbf{y}X \longrightarrow \mathbf{y}Y$$

is induced by some (non–unique) map $X \longrightarrow Y$.

Remark 0.3. This theorem is usually referred to as "Brown representability". The reason for this is that, 10 years earlier, Brown [13] proved a special case. In Brown's theorem, there was a countability hypothesis on the functor.

Calling this theorem "Brown representability" is somewhat confusing, since in the same paper, Brown proved another result, somewhat related. He showed that, if $\mathcal{T}$ is the homotopy category of spectra, and $H : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ is a homological functor taking coproducts to products, then H is representable. There are two theorems here, one about homological functors on $\mathcal{T}^{op}$, and another about homological functors on the subcategory $\{\mathcal{T}^c\}^{op}$. And both theorems usually go under the name Brown representability. Neither theorem is a special case of the other. In the literature, one sometimes distinguishes them by calling the theorem about functors on $\{\mathcal{T}^c\}^{op}$ "Brown representability for homology",

while the theorem about functors on $\mathcal{T}^{op}$ goes by the name "Brown representability for cohomology".

The reason for this strange terminology is the following. In many interesting cases, the category $\mathcal{T}^c$ is self dual. Thus functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ are the same as functors $\mathcal{T}^c \longrightarrow \mathcal{A}b$, and these correspond to functors $\mathcal{T} \longrightarrow \mathcal{A}b$ respecting coproducts. Thus "Brown representability for homology" may be viewed as a theorem about covariant homological functors $\mathcal{T} \longrightarrow \mathcal{A}b$, respecting coproducts.

In hindsight, it seems natural to ask how these statements generalise to other triangulated categories, in particular, the derived category of a ring. Surprisingly, questions of this sort were not asked until the 1980's.

Even then, the first questions to be asked were: To what extent can results about rings be generalised to homotopy theory. The first to suggest that this might be a fruitful pursuit was probably Waldhausen. Waldhausen proposed that techniques from homological algebra—Hochschild homology and cohomology, trace maps, and cyclic versions of these—should all be done in the context of E^∞ ring spectra. The work that followed, by Goodwillie, Bökstedt, Hsiang, Madsen and many others since, showed how good the idea was.

The idea that translating results from homotopy theory to derived categories could be worthwhile came later. The first paper we are aware of is Hopkins' [20]; in it, one has a derived category version of the nilpotence theorem. But it was really only in Bökstedt–Neeman's [11] that the first attempt was made, to use homotopy theoretic techniques to solve standard problems on derived categories. In the 1990's, we have seen explosive growth in the subject. In [34] and [35], Neeman applied techniques coming from homotopy theory to the study of, respectively, the localisation theorem in K–theory and to Grothendieck duality. The articles by Rickard [40], Benson, Carlson and Rickard [7], [8], [9], Benson and Krause [10], Krause [29], and Benson and Gnacadja [6], give beautiful applications to group cohomology. Keller [24], [26], [25] applies the techniques to the study of cyclic homology. And Voevodsky [46], [47] and [45], Suslin–Voevodsky [43] and Morel [31] and [32], have produced a string of results, which apply homotopy theory to the study of motives.

Along with the applications, came the study of the degree to which the theorems extend. Homotopy theorists, over a period of 30 years, developed certain tools to handle the category of spectra. It became interesting to know which parts of these tools work, in the new and greater generality. This has also led to a series of papers. Hovey, Palmieri and Strickland [21] set up a convenient axiomatic formalism. Without going into detail, we remind the reader of the work of Beligiannis [5], Christensen [14], Christensen–Strickland [15], Franke [16], Keller [23], Krause [28], [27], [29], Krause and Reichenbach [30], and Neeman [36], [37] and [38].

This skimpy historical survey was intended to explain why people have studied whether Brown representability generalises to derived categories. As we mentioned in Remark 0.3, the term Brown representability is used to cover two theorems. Brown representability

for cohomology is a characterisation of representable functors $\mathcal{T}^{op} \longrightarrow \mathcal{A}b$, while Brown representability for homology is a more complicated statement about functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$. Of these, the generalisation of Brown representability for cohomology is very well understood. The best and most recent results were obtained independently by Franke [16] and Neeman [38], and one of the remarkable aspects of their theorems, is that they prove new results even in homotopy theory. The theorems tell us, that Brown representability for cohomology generalises to the categories of E–acyclic spectra and E–local spectra, for any homology theory E.

This paper addresses the less well understood problem, of Brown representability for homology. In the remainder of the Introduction, we will do two things. First, we will go through the history of this problem in detail, explaining what was already known. Then, we will outline the counterexamples and results obtained in this article. But before we start, we need to establish some notation.

Notation 0.4. All rings will be associative, with unit. All R–modules will be right, unitary modules. The ring R is called *hereditary* if its global dimension is at most 1. The triangulated category $\mathcal{T} = D(R)$ will be the unbounded derived category of right R–modules. The category $\mathcal{T}^c$ is, as above, the full subcategory of compact objects in $\mathcal{T}$.

We will denote the category of right R–modules by the symbol Mod-R. The subcategory of finitely presented R–modules will be denoted mod-R. The category of all additive functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ will be denoted Mod-$\mathcal{T}^c$, while the category of all additive functors $\{\text{mod-}R\}^{op} \longrightarrow \mathcal{A}b$ will bear the name Mod(mod-R).

When speaking of objects of the category Mod-$\mathcal{T}^c$, that is, of functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$, we frequently wish to single out the ones that are homological, that is, take triangles to long exact sequences. We will feel free to interchangeably use the adjectives "homological", "exact" or "flat". We remind the reader that an object of Mod-$\mathcal{T}^c$ is exact if and only if it is a filtered colimit of representable functors. Furthermore, the representable functors are projective. (We use the term "representable" to mean functors of the form $\mathbf{y}C$, with C compact. In the literature, people sometimes call all functors $\mathbf{y}X$ representable.)

We also need to recall the notion of purity for R–modules. A short exact sequence of R–modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called *pure exact*, if it remains exact when tensored with an arbitrary left R–module. Equivalently, it is a pure exact sequence if, for every finitely presented module P, the functor $\text{Hom}(P, -)$ takes it to an exact sequence

$$0 \longrightarrow \text{Hom}(P, A) \longrightarrow \text{Hom}(P, B) \longrightarrow \text{Hom}(P, C) \longrightarrow 0.$$

An R–module P is called *pure projective*, if the functor $\text{Hom}(P, -)$ takes pure exact sequences to exact sequences. A module P is pure projective if and only if it is a summand of a coproduct of finitely presented modules. The *pure projective dimension*

of an R–module M is defined to be the length of its shortest pure resolution by pure projectives.

A module I is called *pure injective,* if the functor $\mathrm{Hom}(-, I)$ takes pure exact sequences to exact sequences. The pure injective dimension of a module I is the length of the shortest pure resolution by pure injectives. The pure global dimension of R, denoted pgldim R, is the supremum over all M, of the pure projective dimension of M. This equals the supremum of the pure injective dimensions. We refer the reader to [22] for a more thorough discussion, with proofs.

Finally, recall our shorthand: for $X \in \mathcal{T}$, we write $\mathbf{y}X$ for the exact=homological=flat functor $\mathcal{T}(-, X)\big|_{\mathcal{T}^c}$. It is also convenient to make a definition which is not so standard:

Definition 0.5. (Beligiannis [5]**)** *The pure global dimension of $\mathcal{T}$, denoted* pgldim $\mathcal{T}$*, is defined to be the supremum, over all $X \in \mathcal{T}$, of the projective dimension in* Mod-$\mathcal{T}^c$ *of the object $\mathbf{y}X$.*

The following proposition will be useful.

Proposition 0.6. (Beligiannis [5, Prop. 11.2]. The proof is based on an idea by Jensen, which appeared in a paper by Simson [42, Thm. 2.7].)
The pure global dimension of $\mathcal{T}$ is also the supremum over all homological=exact functors F, of the projective dimension of F. Note that, as we will discover in this article, there can be more F's than $\mathbf{y}X$'s.

Let $\mathcal{T}$ be a triangulated category with coproducts, and $\mathcal{T}^c \subset \mathcal{T}$ the full subcategory of compact objects. We adopt the following notation:

[BRO]: The category $\mathcal{T}$ satisfies [BRO] if every exact functor $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ is of the form $\mathbf{y}X$, for some $X \in \mathcal{T}$.

[BRM]: The category $\mathcal{T}$ satisfies [BRM] if every natural transformation $\mathbf{y}X \longrightarrow \mathbf{y}X'$ is induced by a map $X \longrightarrow X'$.

The theorem of Adams (see 0.2) says, that if $\mathcal{T}$ is the homotopy category of spectra, then both [BRO] and [BRM] hold in $\mathcal{T}$. In [36], Neeman found a necessary and sufficient condition for this to generalise, to arbitrary compactly generated $\mathcal{T}$'s. For this article, in the statements that follow, assume $\mathcal{T} = D(R)$ is the derived category of a ring R.

Theorem 0.7. [36] *The following are equivalent:*

(i) *Both [BRM] and [BRO] hold in $\mathcal{T}$*
(ii) pgldim $\mathcal{T} \leq 1$.

The direction (i)$\Longrightarrow$(ii) was also observed in [15]. Beligiannis, using his Proposition 0.6 above, recently showed:

Theorem 0.8. [5, Theorem 11.8] *[BRM]$\Longrightarrow$[BRO].*

Neeman [36] also showed that when R is countable, [BRM] (and therefore also [BRO]) holds.

Keller produced the first example, where [BRM] fails. It may be found in Neeman's [36]. The example hinges on the following observation. If [BRM] holds, then by Theorem 0.7, we have pgldim $\mathcal{T} \leq 1$. That is, for any object $X \in \mathcal{T}$, $\mathbf{y}X$ has projective dimension at most 1. If R is a noetherian ring, this means that the cohomology modules H^iY have pure projective dimension at most 1. For a counterexample, one needs only produce an object $Y \in \mathcal{T} = D(R)$, so that its cohomology is of pure projective dimension greater than 1.

The most recent progress preceding this article is a theorem of Beligiannis:

Theorem 0.9. [5, Remark 11.12] *[BRO] holds, whenever* pgldim $\mathcal{T} \leq 2$.

This leaves several obvious questions:

Q1: What is the precise relation between the pure global dimension of R, denoted pgldim R, and the pure global dimension of $\mathcal{T}$, denoted pgldim $\mathcal{T}$?

Q2: Just how closely are the two related to [BRM] and [BRO]?

Q3: Does [BRO] hold in general?

In this article, we make progress on these questions. Regarding Q1, we prove that for many rings pgldim $R \leq$ pgldim $\mathcal{T}$, and that for hereditary rings this is an equality. Then we give examples to show that in general the inequality can be strict.

Regarding Q2, we give a precise relationship between pure global dimension, [BRO] and [BRM] for hereditary rings. Then we give examples to show that in general no such simple relationship holds. At the same time we show that [BRO] can fail, answering Q3. For example, it fails for $R = k[x, y]$ when k has cardinality at least $\aleph_3$ (Example 2.12).

Here is a more detailed overview of these results. We begin with an easy proposition giving our positive results about Q1. It is followed by a description of our counterexamples. We end with our positive results about Q2.

Proposition 1.4

(i) *Suppose that R is a coherent ring, and that all finitely presented R–modules are of finite projective dimension. (This hypothesis holds when R is noetherian of finite global dimension.) Then we have*

$$\text{pgldim } R \leq \text{pgldim } D(R).$$

(ii) *Suppose that R is hereditary. Then we have*

$$\text{pgldim } R = \text{pgldim } D(R).$$

Weaker versions of this proposition were known before, and the inequality was after all at the basis of Keller's counterexample to [BRM]. The really new result we show in this article is that, for some R, the inequality can be strict; Example 1.5 gives such an R. The idea of the counterexample is to produce two rings R and S, of different pure global dimensions, but with $D(R) \cong D(S)$. Then pgldim $D(R) =$ pgldim $D(S)$ must be at least the maximum, and strictly bigger than the minimum, of pgldim R and pgldim S. These rings are finite-dimensional non-commutative k-algebras described by means of quivers.

Even more, we show that in general the answer to Q3 is negative: [BRO] can fail. It fails for the rings R and S mentioned above when the cardinality of k is at least $\aleph_2$, for the ring $k[x,y]$ when $|k| \geq \aleph_3$, and also for the ring $T = k\langle X, Y\rangle$ of polynomials in two non-commuting variables when $|k| \geq \aleph_2$. (In particular, since it is consistent with ZFC that $|\mathbb{C}| = \aleph_3$, it is impossible to prove [BRO] using ZFC when $R = \mathbb{C}[x,y]$.) The proof that these are counterexamples is presented in Section 2. Our method is to find an exact sequence

$$0 \longrightarrow \mathbf{y}A \longrightarrow F \longrightarrow \mathbf{y}B \longrightarrow 0$$

in Mod-$\mathcal{T}^c$, and show that F is not isomorphic to $\mathbf{y}Y$ for any Y. The idea is to study the extension group $\mathrm{Ext}^1(\mathbf{y}B, \mathbf{y}A)$. We get a handle on this group using several spectral sequences. The precise statement of our theorem is:

Theorem 2.11. *Let R be an associative ring. Assume that R is coherent, and that every finitely presented R–module has a finite projective resolution. Suppose there exists an R–module N so that*

$$\text{pure inj dim}(N) - \text{inj dim}(N) \geq 2.$$

Then [BRO] fails for in $D(R)$. This means that there exists a homological functor $F : \{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$, which is not the restriction of any representable. That is, there exists no Y with $\mathbf{y}Y = F$.

What is mysterious here, is that given a homological F, we cannot directly tell whether it is of the form $\mathbf{y}X$. We have no criterion to distinguish $\mathbf{y}X$'s from other homological functors. In fact, Beligiannis' Proposition 0.6 tells us, that given any homological F, there exists a $\mathbf{y}X$ of projective dimension greater than or equal to that of F; projective dimension will not distinguish $\mathbf{y}X$'s from other homological functors. What we do amounts to finding a trick, to get around this problem.

For general rings, this is all we can say. We can give a refinement of the results for hereditary rings R; recall that R is *hereditary* if its global dimension is ≤ 1. Examples of hereditary rings are commutative principal ideal domains, and non-commutative polynomial rings.

Theorem 2.13. *Let R be a hereditary ring. Then*

(i) *[BRM] holds in $\mathcal{T}$ if and only if the pure global dimension of R is at most 1; and*
(ii) *[BRO] holds in $\mathcal{T}$ if and only if the pure global dimension of R is at most 2.*

We conclude the paper with the observation (Lemma 2.14) that any counterexample to [BRO] must take values in infinite-dimensional vector spaces.

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1. Pure global dimension: module categories versus derived categories

Let R be an associative ring. We denote by $\mathcal{T}$ the unbounded derived category $D(R)$ of the category of (right) R–modules, and by $\mathcal{T}^c$ the full subcategory of compact objects. Recall that a complex is a compact object of $\mathcal{T}$ iff it is quasi-isomorphic to a bounded complex of finitely generated projective R–modules. Here and elsewhere, we identify the category Mod-R of R–modules with the subcategory of $\mathcal{T}$ consisting of complexes concentrated in degree 0.

Lemma 1.1. *The following are equivalent:*

(i) *R is coherent and each finitely presented R–module is of finite projective dimension.*
(ii) *Each finitely presented R–module is compact when viewed as an object of $D(R)$.*
(iii) *A complex X is compact iff each H^nX is finitely presented and $H^nX \cong 0$ for all but finitely many n.*

Remark 1.2. In particular, the conditions of the lemma are satisfied if R is noetherian and of finite global dimension. They are also satisfied by any hereditary ring, that is, any ring of global dimension at most 1.

Proof. We will prove (i)$\Longleftrightarrow$(ii), and then that (i)+(ii)$\Longleftrightarrow$(iii). But first, we remind the reader that a ring is coherent iff the kernel of every map between finitely generated projective modules is finitely presented. We will also use the easy fact that a module is a compact object of $D(R)$ iff it admits a finite resolution by finitely generated projective objects.

Assume (i) holds. Let M be a finitely presented module. Since R is coherent, M admits a resolution by finitely generated projective modules. Since M is of finite projective dimension, this resolution may be chosen to be finite. So M is compact in $D(R)$. That is, (ii) follows.

Suppose that (ii) holds. Then each finitely presented module admits a finite resolution by finitely generated projectives, and so in particular has finite projective dimension. Now let K be the kernel of a map $f : P_1 \longrightarrow P_0$ between finitely generated projectives. Let C be the cokernel of f. In $D(R)$, we have the canonical triangle

$$\Sigma K \longrightarrow P \longrightarrow C \longrightarrow \Sigma^2 K,$$

where P is the complex $P_1 \longrightarrow P_0$. By assumption, P and C are compact. Hence K is compact. So it admits a finite resolution by finitely generated projective objects. In particular, it is finitely presented. Thus R is coherent; (i) holds.

Thus far, we have proved (i)$\Longleftrightarrow$(ii). Assume these equivalent conditions hold; we wish to prove (iii). Let X be a compact object in $D(R)$. It is isomorphic to a finite complex of finitely generated projective modules. By (i), R is coherent; hence H^nX is finitely presented for all n. And since the complex X is finite, $H^nX \cong 0$ for all but finitely many n.

Suppose now that H^nX is finitely presented for all n, and that $H^nX \cong 0$ for all but finitely many n. The t–structure on $D(R)$ gives us triangles

$$X^{\leq n} \longrightarrow X \longrightarrow X^{>n} \longrightarrow \Sigma X^{\leq n}$$

and these allow us to assemble X from its homology. Now H^nX is finitely presented for all n, and by (ii) it is compact. This forces X, an iterated extension of compact objects, to also be compact. We conclude that (iii) holds.

Finally, (iii)$\Longrightarrow$(ii) is immediate. □

Recall that the functor $\mathbf{y} : \mathcal{T} \to \text{Mod-}\mathcal{T}^c$ sends an object $X \in \mathcal{T}$ to the functor

$$\mathbf{y}X = \mathcal{T}(-, X)|_{\mathcal{T}^c}.$$

For $i \in \mathbb{Z}$ and $F \in \text{Mod-}\mathcal{T}^c$, we define the i^{th} *homology of* F by

$$H^iF = F(\Sigma^{-i}R).$$

The functor $H^i : \text{Mod-}\mathcal{T}^c \longrightarrow \text{Mod-}R$ extends the homology functor on $\mathcal{T}$ in the sense that we have a canonical isomorphism $H^i \circ \mathbf{y} = H^i$.

An object G in the category $\text{Mod-}\mathcal{T}^c$ is called *finitely presented,* if there exists an exact sequence

$$\mathbf{y}X \longrightarrow \mathbf{y}Y \longrightarrow G \longrightarrow 0.$$

The full subcategory of all finitely presented objects in $\text{Mod-}\mathcal{T}^c$ is known to be an abelian category; see for example Freyd's [17]. As in the case of a module category, a sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

of $\text{Mod-}\mathcal{T}^c$ is called *pure exact* if the sequence

$$0 \longrightarrow \text{Hom}(G, F_1) \longrightarrow \text{Hom}(G, F_2) \longrightarrow \text{Hom}(G, F_3) \longrightarrow 0$$

is exact for each finitely presented functor G. (In particular, the sequence is then exact.)

Lemma 1.3. *Suppose that the conditions of Lemma 1.1 hold.*

(i) *The functor* $\mathbf{y} : \text{Mod-}R \longrightarrow \text{Mod-}\mathcal{T}^c$ *commutes with filtered colimits. It takes pure projective* R*–modules to projective objects of* $\text{Mod-}\mathcal{T}^c$*. It transforms pure exact sequences of* R*–modules into pure exact sequences in* $\text{Mod-}\mathcal{T}^c$*.*

(ii) *For each* $i \in \mathbb{Z}$*, the functor* H^i *commutes with filtered colimits. It takes projective objects of* $\text{Mod-}\mathcal{T}^c$ *to pure projective* R*–modules. It transforms pure exact sequences of* $\text{Mod-}\mathcal{T}^c$ *into pure exact sequences of* R*–modules.*

Proof. (i) Let M_λ be a filtered system of R–modules. Clearly, if $P = \Sigma^i R$ for some $i \in \mathbb{Z}$, the canonical map

$$\varinjlim \mathcal{T}(P, M_\lambda) \longrightarrow \mathcal{T}(P, \varinjlim M_\lambda)$$

is bijective. Since both sides are cohomological functors of P, this map is still bijective if P is any compact object of $\mathcal{T}$, since $\mathcal{T}^c$ is the thick subcategory generated by R. This means that $\mathbf{y}$ takes $\operatorname{colim} M_\lambda$ to $\operatorname{colim} \mathbf{y}M_\lambda$.

Each pure projective R–module is a direct factor of a coproduct of finitely presented modules. Since the functor $\mathbf{y}$ commutes with coproducts, it is enough to show that $\mathbf{y}M$ is projective if M is finitely presented. But in this case, M is compact in $\mathcal{T}$, by our assumption on the ring R. So $\mathbf{y}M$ is projective since it is even representable.

Now let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a pure exact sequence of R–modules. Clearly, if N is finitely presented, the sequence splits. An arbitrary module N is a filtered colimit of finitely presented modules. Thus the sequence is a filtered colimit of split sequences. Since the functor $\mathbf{y}$ commutes with filtered colimits, the image of the sequence is also a filtered colimit of split sequences. Thus it is pure.

(ii) By definition, the functor H^i is evaluation at $\Sigma^{-i}R$. Thus it commutes with colimits. The projective objects of Mod-$\mathcal{T}^c$ are direct factors of coproducts of representable functors, and the functor H^i commutes with coproducts. So it is enough to show that $H^i\mathbf{y}P = H^iP$ is pure projective for $P \in \mathcal{T}^c$. This is clear since H^iP is finitely presented, by our assumption on the ring R.

Let

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

be a pure exact sequence of Mod-$\mathcal{T}^c$. Clearly if F_3 is finitely presented, the sequence splits. In the general case, F_3 is a filtered colimit of a system of finitely presented functors. So the sequence is a filtered colimit of split sequences. Since the functor H^i commutes with filtered colimits, this implies the last assertion. □

The *pure global dimension* of the derived category $D(R) = \mathcal{T}$ is by definition [5] the supremum of the projective dimensions of the functors $\mathbf{y}X$, $X \in \mathcal{T}$. We write pgldim for 'pure global dimension'. Part (ii) of the following lemma is due to Beligiannis [5, Prop. 12.8].

Proposition 1.4. *Suppose that the conditions of Lemma 1.1 hold.*

(i) *Let M be an R–module. Then the projective dimension of $\mathbf{y}M$ equals the pure projective dimension of M. Hence we have*

$$\text{pgldim } R \leq \text{pgldim } D(R).$$

(ii) *Suppose that R is hereditary. Then we have*

$$\text{pgldim } R = \text{pgldim } D(R).$$

Proof. (i) The first part of the preceding lemma shows that the functor $\mathbf{y}$ takes pure projective resolutions of a module M to projective resolutions of $\mathbf{y}M$. Hence the projective dimension of $\mathbf{y}M$ is no more than the pure projective dimension of M. Conversely, let

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbf{y}M \longrightarrow 0$$

be a projective resolution of $\mathbf{y}M$. If M is finitely presented, then $\mathbf{y}M$ is projective, so the resolution is nullhomotopic. An arbitrary M is still a filtered colimit of finitely presented modules. So for arbitrary M the resolution is a filtered colimit of nullhomotopic complexes. Thus it is a pure exact sequence. By the second part of the above lemma, its image under H^0 is a pure projective resolution of $H^0\mathbf{y}M = M$. Thus the pure projective dimension of M is no more than the projective dimension of $\mathbf{y}M$.

(ii) By part (i), it suffices to prove that pgldim $R \geq$ pgldim $D(R)$. Let $X \in D(R)$. Since R is hereditary, the object X is isomorphic in $D(R)$ to the coproduct of the $\Sigma^{-i}H^iX$, $i \in \mathbb{Z}$; see Lemma 6.7, on page 153 of [33]. Hence the projective dimension of $\mathbf{y}X$ is no greater than the supremum of the projective dimensions of the $\mathbf{y}H^iX$. These are bounded by pgldim R thanks to part (i). □

Example 1.5. Let k be a field and let t be the cardinal such that $\aleph_t = \max(|k|, \aleph_0)$. So t is 0 if k is finite or countable, 1 if k has the smallest uncountable cardinality, etc. We will exhibit a k-algebra R such that the inequality

$$\text{pgldim } R \leq \text{pgldim } D(R)$$

is strict. Our example is based on the observation that there are algebras with equivalent derived categories but widely differing pure global dimensions. More precisely, we will exhibit a finite-dimensional k-algebra R with pgldim $R = 0$ such that $D(R)$ is triangle equivalent to $D(S)$ for a finite-dimensional hereditary k-algebra S whose pure global dimension is $t + 1$ (∞ if t is infinite). Thus we have

$$\text{pgldim } R < \text{pgldim } S = \text{pgldim } D(S) = \text{pgldim } D(R),$$

where we have used part (ii) of the above proposition for the first equality.

Thus Theorem 2.13 implies that [BRM] fails for $D(R)$ when $t \geq 1$ and that [BRO] fails for $D(R)$ when $t \geq 2$, even though R has pure global dimension 0.

We will define the algebras R and S using the language of quivers with relations (cf. [41], [19], [3]). Here is all we need: A *quiver* is an oriented graph. It is thus given by a set Q_0 of points, a set Q_1 of arrows, and two maps $s, t : Q_1 \longrightarrow Q_0$ associating with each arrow its source and its target. A simple example is the quiver

$$\vec{A}_{10} \ : \quad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \longrightarrow \cdots \longrightarrow 8 \xrightarrow{\alpha_8} 9 \xrightarrow{\alpha_9} 10.$$

A *path* in a quiver Q is a sequence $(y|\beta_r|\beta_{r-1}|\cdots|\beta_1|x)$ of composable arrows β_i with $s(\beta_1) = x$, $s(\beta_i) = t(\beta_{i-1})$, $2 \leq i \leq r$, $t(\beta_r) = y$. In particular, for each point $x \in Q_0$, we have the *lazy path* $(x|x)$. It is neutral for the obvious *composition* of paths. The *quiver algebra* kQ has as its basis all paths of Q. The product of two basis elements equals the composition of the two paths if they are composable and 0 otherwise. For example,

the quiver algebra of $Q = \vec{A}_{10}$ is isomorphic to the algebra of lower triangular 10×10 matrices.

The construction of the quiver algebra kQ is motivated by the (easy) fact that the category of left kQ-modules is equivalent to the category of all diagrams of vector spaces of the shape given by Q. It is not hard to show that each quiver algebra is hereditary. It is finite-dimensional over k iff the quiver has no oriented cycles.

Gabriel [18] showed that the quiver algebra of a finite quiver has only a finite number of k–finite-dimensional indecomposable modules (up to isomorphism) iff the underlying graph of the quiver is a disjoint union of Dynkin diagrams of type A, D, E.

The above example has underlying graph of Dynkin type A_{10} and thus its quiver algebra has only a finite number of finite-dimensional indecomposable modules.

An ideal I of a finite quiver Q is *admissible* if for some N we have

$$(kQ_1)^N \subseteq I \subseteq (kQ_1)^2,$$

where (kQ_1) is the two-sided ideal generated by all paths of length 1. A *quiver Q with relations R* is a quiver Q with a set R of generators for an admissible ideal I of kQ. The algebra kQ/I is then the *algebra associated with* (Q, R). Its category of left modules is equivalent to the category of diagrams of vector spaces of shape Q obeying the relations in R. The algebra kQ/I is finite-dimensional (since I contains all paths of length at least N), hence artinian and noetherian. By induction on the number of points one can show that if the quiver Q contains no oriented cycle, then the algebra kQ/I is of finite global dimension.

One can show that every finite-dimensional algebra over an algebraically closed field is Morita equivalent to the algebra associated with a quiver with relations and that the quiver is unique (up to isomorphism).

Now we let R be the finite-dimensional k-algebra associated with the above quiver $\vec{A}_{10}$ and the relation $\alpha_8\alpha_7\cdots\alpha_1$ (no α_9!). The algebra R is a quotient of $k\vec{A}_{10}$ and thus it admits only a finite number of indecomposable finite-dimensional modules. By a result of Auslander [2] and Tachikawa [44], this is equivalent to pgldim $R = 0$.

Let S be the quiver algebra of the quiver

$$\begin{array}{rccccccccccc} & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 7 & \longrightarrow & 8 & \longrightarrow & 9 & \longrightarrow & 10 \\ E: & & & & & & & & & & & & & \downarrow & & & & \\ & & & & & & & & & & & & & 1. & & & & \end{array}$$

Thus S is finite-dimensional over k and hereditary. By Theorem 4.1 of Baer-Lenzing's [4], we have pgldim $S = t + 1$ (∞ if t is infinite).

Finally, we need to show that R and S have equivalent derived categories. Indeed, the algebra R admits a tilting complex with endomorphism ring S so that the equivalence follows from Rickard's Morita theorem for derived categories [39]. To describe the tilting complex, let $P_i = e_iR$ be the projective R–module associated with the idempotent $e_i = (i|i)$ (the lazy path). It is easy to compute the morphism spaces between these modules: Indeed, we have $\mathrm{Hom}(e_iR, e_jR) = e_jRe_i$ and this space identifies with the vector space

on the set of paths from i to j divided by the subspace of linear combinations of paths lying in the ideal of relations. For example, for $i \leq j$, the path from i to j yields a canonical morphism $P_i \longrightarrow P_j$, which vanishes iff $(i,j) = (1,9)$ or $(i,j) = (1,10)$. The tilting complex T is now the sum of the complexes

$$T_2 = (P_1 \longrightarrow P_2), \quad T_3 = (P_1 \longrightarrow P_3), \quad \dots \ , \quad T_8 = (P_1 \longrightarrow P_8),$$
$$T_1 = (P_1 \longrightarrow 0), \quad T_9 = (0 \longrightarrow P_9), \quad T_{10} = (0 \longrightarrow P_{10}),$$

where the first term of each complex is in degree 0. Using the description of the morphism spaces between the P_i it is not hard to check that, in the homotopy category of right R-modules, we do have $\mathrm{Hom}(T_i, T_j[l]) = 0$ for all i, j and all $l \neq 0$, and that the endomorphism ring of T is indeed isomorphic to S. For example, the canonical idempotent $(i|i)$ of the quiver E corresponds to the idempotent of $\mathrm{End}(T)$ arising from the identity of T_i and the arrow $8 \longrightarrow 9$ of E corresponds to the obvious morphism of complexes

$$\begin{array}{ccc} P_1 & \longrightarrow & P_8 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & P_9 \end{array}$$

which is well-defined thanks to the relation $\alpha_8\alpha_7 \cdots \alpha_1$ that we imposed.

2. Failure of Brown representability

In this section, R will be a ring satisfying the equivalent conditions of Lemma 1.1. In particular, all the theorems hold if R is a noetherian ring of finite global dimension, or if R is hereditary. We begin by reminding ourselves of a standard spectral sequence.

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category satisfying AB5, and with enough projectives. Suppose that X and Y are objects of $\mathcal{A}$ and that $X = \varinjlim X_\lambda$ expresses X as a filtered colimit of objects $X_\lambda \in \mathcal{A}$. Then there is a spectral sequence, converging to $\mathrm{Ext}^{i+j}(X,Y)$, whose E^2 term is*

$$\varprojlim{}^i \mathrm{Ext}^j(X_\lambda, Y).$$

Proof. There is a standard chain complex which computes the derived functors of $\varinjlim$. Since the abelian category $\mathcal{A}$ satisfies AB5, the derived functors of filtered colimits vanish, and we deduce an exact sequence in $\mathcal{A}$

$$\cdots \longrightarrow \bigoplus_{\lambda \to \mu} X_\lambda \longrightarrow \bigoplus_{\lambda} X_\lambda \longrightarrow X \longrightarrow 0.$$

This gives us a resolution of X in $\mathcal{A}$, and the spectral sequence is just the spectral sequence of the functor $\mathrm{Ext}^*(-, Y)$ applied to this resolution. □

In the following, we write mod-R for the category of finitely presented R–modules and Mod(mod-R) for the category of contravariant additive functors from mod-R to $\mathcal{A}b$. The

object

$$\text{Mod-}R\left(-\,,\,M\right)\Big|_{\text{mod-}R}$$

of Mod(mod-R) will be denoted $\mathbf{z}M$.

Lemma 2.2. *Let R be a ring, and let M_λ be a filtered diagram of R–modules with colimit M. Then*

(i) $\mathbf{y}M = \operatorname{colim}\limits_{\longrightarrow} \mathbf{y}M_\lambda$ *in* Mod-$\mathcal{T}^c$.

(ii) $\mathbf{z}M = \operatorname{colim}\limits_{\longrightarrow} \mathbf{z}M_\lambda$ *in* Mod(mod-R).

Proof. (i) was proved in Lemma 1.3 (i). The second statement is more familiar in the equivalent form, which states that $\text{Mod-}R(K, M) = \operatorname{colim}\limits_{\longrightarrow} \text{Mod-}R(K, M_\lambda)$ for any finitely presented K. This is not hard to prove. □

Reminder 2.3. Let R be a ring and let M be an R–module. Consider the filtered diagram of finitely presented modules M_λ equipped with a map to M. Then M is the colimit of this diagram; we already used this in the proof of Proposition 1.4(i). This is the setting in which we will apply Lemma 2.2.

The following lemma is well known; the proof may be found, for example, in Theorem 2.8 of Simson's [42]. We include a sketch of the proof for the reader's convenience.

Lemma 2.4. *Let R be a ring satisfying the conditions of Lemma 1.1, and let M be an R–module. As mentioned in Remark 2.3, M is the filtered colimit of all finitely presented modules M_λ mapping to M.*

(i) *Let F be an object of* Mod-$\mathcal{T}^c$. *That is, F is a functor $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$. Then the group* $\operatorname{Ext}^i(\mathbf{y}M, F)$ *of extensions in* Mod-$\mathcal{T}^c$ *is isomorphic to* $\varprojlim^i F(M_\lambda)$.

(ii) *Let F be an object of* Mod(mod-R). *That is, F is a functor* $\{\text{mod-}R\}^{op} \longrightarrow \mathcal{A}b$. *Then the group* $\operatorname{Ext}^i(\mathbf{z}M, F)$ *of extensions in* Mod(mod-R) *is isomorphic to* $\varprojlim^i F(M_\lambda)$.

Proof. (i): By Lemma 2.2, $\mathbf{y}M$ is the colimit of $\mathbf{y}M_\lambda$ in Mod-$\mathcal{T}^c$. Lemma 2.1 then tells us that we get a spectral sequence with E^2 term

$$\varprojlim^i \operatorname{Ext}^j(\mathbf{y}M_\lambda, F)$$

converging to the group $\operatorname{Ext}^{i+j}(\mathbf{y}M, F)$ of extensions in Mod-$\mathcal{T}^c$. The functor $\mathbf{y}M_\lambda$ is representable, since by our hypothesis on R the module M_λ is compact. Thus $\mathbf{y}M_\lambda$ is projective, the Ext^j terms vanish unless $j = 0$, the spectral sequence collapses, and the desired isomorphism follows.

The proof of (ii) is similar. □

Remark 2.5. In part (i) of Lemma 2.4, we computed the extensions of $\mathbf{y}M$ by F. This interests us most in the case where $F = \mathbf{y}\Sigma^j N$, with N an R–module. In this case, the

computation tells us that we have isomorphisms

$$\mathrm{Ext}^i(\mathbf{y}M, \mathbf{y}\Sigma^j N) = \varprojlim{}^i\, \mathcal{T}(M_\lambda, \Sigma^j N) = \varprojlim{}^i\, \mathrm{Ext}^j_R(M_\lambda, N).$$

In part (ii) of Lemma 2.4, we computed the extensions of $\mathbf{z}M$ by F. This interests us most in the case where $F = \mathbf{z}N$, with N an R–module. In this case, the computation tells us that we have an isomorphism

$$\mathrm{Ext}^i(\mathbf{z}M, \mathbf{z}N) = \varprojlim{}^i\, \mathrm{Hom}_R(M_\lambda, N).$$

Moreover the group $\mathrm{Ext}^i(\mathbf{z}M, \mathbf{z}N)$ of extensions in $\mathrm{Mod}(\text{mod-}R)$ can be identified with the group $\mathrm{PExt}^i(M, N)$; see [22]. We deduce that

$$\mathrm{PExt}^i(M, N) = \varprojlim{}^i\, \mathrm{Hom}_R(M_\lambda, N).$$

Corollary 2.6. *If M and N are R–modules and $j > 0$, then every map $\mathbf{y}\Sigma^j M \longrightarrow \mathbf{y}N$ vanishes. Moreover, maps $\mathbf{y}M \longrightarrow \mathbf{y}N$ are in one-to-one correspondence with maps of R–modules $M \longrightarrow N$.*

Proof. For $j > 0$, we must show that any map $\mathbf{y}M \longrightarrow \mathbf{y}\Sigma^{-j}N$ vanishes. But by Remark 2.5, the group of such maps is

$$\varprojlim{}^0\, \mathrm{Ext}^{-j}_R(M_\lambda, N),$$

which vanishes because there are no extensions of negative degree.

The group of maps $\mathbf{y}M \longrightarrow \mathbf{y}N$ is exactly

$$\varprojlim{}^0\, \mathrm{Ext}^0_R(M_\lambda, N),$$

which is $\mathrm{Hom}_R(M, N)$. □

Lemma 2.7. *Let F be an object in* Mod-$\mathcal{T}^c$*, that is, a contravariant additive functor from $\mathcal{T}^c$ to $\mathcal{A}b$. Suppose there exists an integer $j > 0$, R–modules M and N, and a short exact sequence in* Mod-$\mathcal{T}^c$

$$0 \longrightarrow \mathbf{y}\Sigma^j N \xrightarrow{\alpha} F \xrightarrow{\beta} \mathbf{y}M \longrightarrow 0.$$

Then this sequence is unique up to isomorphism.

Proof. The integer j and the modules M and N are clearly determined by the homology of F. In Corollary 2.6 we saw that any map $\mathbf{y}\Sigma^j N \longrightarrow \mathbf{y}M$ vanishes. Therefore, given any map $\gamma : \mathbf{y}\Sigma^j N \longrightarrow F$, the composite

$$\mathbf{y}\Sigma^j N \xrightarrow{\gamma} F \xrightarrow{\beta} \mathbf{y}M$$

vanishes, and hence γ must factor through α. Dually, any map $F \longrightarrow \mathbf{y}M$ must factor through β. This shows that the given exact sequence is unique. □

Lemma 2.8. *Let F be an object of* Mod-$\mathcal{T}^c$*, and suppose there exists an integer $j > 0$, R–modules M and N, and a short exact sequence in* Mod-$\mathcal{T}^c$

$$0 \longrightarrow \mathbf{y}\Sigma^j N \xrightarrow{\alpha} F \xrightarrow{\beta} \mathbf{y}M \longrightarrow 0.$$

The functor F will be of the form $\mathbf{y}Y$ if and only if the short exact sequence comes from a triangle. That is, if and only if there exists a triangle in $\mathcal{T}$

$$\Sigma^j N \longrightarrow Y \longrightarrow M \xrightarrow{\partial} \Sigma^{j+1} N$$

with ∂ a phantom map, so that the sequence

$$0 \longrightarrow \mathbf{y}\Sigma^j N \xrightarrow{\alpha} F \xrightarrow{\beta} \mathbf{y}M \longrightarrow 0$$

is obtained by restricting the representable functors to $\mathcal{T}^c$.

We remind the reader that a map $W \longrightarrow X$ in $\mathcal{T}$ is called *phantom* if the composite $C \longrightarrow W \longrightarrow X$ is zero for each compact object C and each map $C \longrightarrow W$.

Proof. The implication $\Longleftarrow$ is trivial. If the triangle exists and is isomorphic to the short exact sequence of functors on $\mathcal{T}^c$, then F is the restriction of a representable functor on $\mathcal{T}$. We wish to prove $\Longrightarrow$. We suppose therefore that the short exact sequence of functors is given, and that F is the restriction of a representable. We want to produce a triangle.

The short exact sequence

$$0 \longrightarrow \mathbf{y}\Sigma^j N \xrightarrow{\alpha} F \xrightarrow{\beta} \mathbf{y}M \longrightarrow 0$$

permits us easily to compute $F(\Sigma^n R)$, for all $n \in \mathbb{Z}$. We have

$$F(\Sigma^n R) \quad = \quad \begin{cases} M & \text{if } n = 0 \\ N & \text{if } n = j \\ 0 & \text{otherwise.} \end{cases}$$

But if $F = \mathbf{y}Y$, then $F(\Sigma^n R) = H^{-n}(Y)$. The above computes for us the cohomology of Y, as an object in $D(R) = \mathcal{T}$.

There is a t-structure truncation on $D(R)$, giving a triangle

$$Y^{\leq -1} \longrightarrow Y \longrightarrow Y^{\geq 0} \xrightarrow{\partial} \Sigma Y^{\leq -1},$$

and our homology computation shows that $Y^{\leq -1}$ and $Y^{\geq 0}$ each have only one non-zero cohomology group. The triangle is therefore of the form

$$\Sigma^j N \longrightarrow Y \longrightarrow M \xrightarrow{\partial} \Sigma^{j+1} N.$$

We deduce an exact sequence

$$\mathbf{y}\Sigma^j N \longrightarrow \mathbf{y}Y \longrightarrow \mathbf{y}M.$$

Now recall that $\mathbf{y}Y = F$, and that by the proof of Lemma 2.7, any map $\mathbf{y}\Sigma^j N \longrightarrow F$ factors through α, and any map $F \longrightarrow \mathbf{y}M$ factors through β. The exact sequence

coming from the triangle therefore factors through

$$\begin{array}{ccccccccc} & & & & \mathbf{y}\Sigma^j N & & & & \\ & & & & \downarrow f & & & & \\ 0 & \longrightarrow & \mathbf{y}\Sigma^j N & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & \mathbf{y}M & \longrightarrow & 0. \\ & & & & & & \downarrow g & & \\ & & & & & & \mathbf{y}M & & \end{array}$$

By Corollary 2.6, the morphisms f and g in the diagram above come from maps of modules $N \longrightarrow N$ and $M \longrightarrow M$. Evaluating the functors at R and $\Sigma^j R$, we compute that both f and g are isomorphisms. Hence the triangle gives rise to the short exact sequence of functors, and ∂ must be a phantom map. □

Next comes a spectral sequence argument. To help the reader, we will first do the easy, baby case.

Proposition 2.9. *Let R be a ring satisfying the conditions of Lemma 1.1. Let N be an R–module with injective dimension at most 1 and pure injective dimension at least 3. Then in* Mod-$\mathcal{T}^c$ *there exists a homological functor $F : \{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ which is not the restriction of any representable. That is, there exists no Y with $\mathbf{y}Y = F$.*

Example 2.10. Let k be a field and R the algebra of the quiver E of Example 1.5 (we called it S there). Then R is finite-dimensional over k and hereditary, since it is the quiver algebra of a finite quiver. So all R–modules are of injective dimension at most 1. Assume that k is infinite of cardinality $\aleph_t$. Then by [4], the pure global dimension of R equals $t+1$ (∞ if t is infinite). Thus when $t \geq 2$ there does exist an R–module satisfying the assumptions of the proposition.

Similarly, the ring $k\langle X, Y\rangle$ of polynomials in two non-commuting variables is an example when $t \geq 2$.

To obtain examples where R is commutative, we will need to use Theorem 2.11, which is a refined version of the above proposition.

Proof. Because N is of pure injective dimension at least 3, there exists a module M and integer $n \geq 3$, so that $\mathrm{PExt}^n(M, N) \neq 0$. If $n > 3$, choose a pure exact sequence

$$0 \longrightarrow M' \longrightarrow P \longrightarrow M \longrightarrow 0,$$

with P pure projective. Then $\mathrm{PExt}^n(M, N) = \mathrm{PExt}^{n-1}(M', N)$. By a sequence of such dimension shifts, we may find an M so that

$$\mathrm{PExt}^3(M, N) \neq 0.$$

By Remark 2.3, we may express M as a filtered colimit of finitely presented modules M_λ. By Lemma 2.1, applied this time to the category of R–modules, there is a spectral

sequence with E^2 term $\varprojlim^i \operatorname{Ext}_R^j(M_\lambda, N)$ converging to $\operatorname{Ext}_R^{i+j}(M, N)$. We will now compute in this spectral sequence.

In Remark 2.5, we computed that

$$\varprojlim{}^3 \operatorname{Ext}_R^0(M_\lambda, N) = \operatorname{PExt}^3(M, N),$$

and by the above, this does not vanish. On the other hand, we know that $\operatorname{Ext}_R^3(M, N) = 0$, since by hypothesis N is of injective dimension at most 1. It follows that one of the differentials in the spectral sequence into the term

$$\varprojlim{}^3 \operatorname{Ext}_R^0(M_\lambda, N)$$

must be non-zero.

But there are only two differentials into this term, one from $\varprojlim^1 \operatorname{Ext}^1$ and one from $\varprojlim^0 \operatorname{Ext}^2$. The latter vanishes, since by hypothesis N is of injective dimension at most 1. It follows that

$$\varprojlim{}^1 \operatorname{Ext}_R^1(M_\lambda, N) \neq 0.$$

But in Lemma 2.4 we showed that this is the group of extensions, in Mod-$\mathcal{T}^c$,

$$0 \longrightarrow \mathbf{y}\Sigma N \longrightarrow F \longrightarrow \mathbf{y}M \longrightarrow 0.$$

The group does not vanish so we may choose a non-trivial extension. Since F is the extension of two homological functors, F must be homological. Now we will show that F cannot be isomorphic to a functor $\mathbf{y}Y$.

Lemma 2.8 tells us that if F is isomorphic to $\mathbf{y}Y$, then there is a triangle in $\mathcal{T}$

$$\Sigma N \longrightarrow Y \longrightarrow M \xrightarrow{\partial} \Sigma^2 N$$

so that the exact sequence of functors above is isomorphic to the one obtained from the triangle. But the map $\partial : M \longrightarrow \Sigma^2 N$ is an element of

$$\operatorname{Ext}^2(M, N) = 0,$$

and therefore the triangle splits. The exact sequence of functors is not split, and we conclude that F cannot be isomorphic to any $\mathbf{y}Y$. □

The next Theorem is the more macho computation with the same spectral sequence.

Theorem 2.11. *Let R be a ring satisfying the conditions of Lemma 1.1. Suppose there exists an R–module N so that*

$$\text{pure inj dim}(N) - \text{inj dim}(N) \geq 2.$$

Then [BRO] fails for in $D(R)$. This means that there exists a homological functor $F : \{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ which is not the restriction of any representable. That is, there exists no Y with $\mathbf{y}Y = F$.

Proof. Let N be a module satisfying the hypotheses. Let $n = \operatorname{inj\,dim}(N)$. Then $\operatorname{pure\ inj\,dim}(N) \geq n+2$. As in the proof of Proposition 2.9, we may choose a module M with $\operatorname{PExt}^{n+2}(M,N) \neq 0$. We may also express M as a filtered colimit of finitely presented modules M_λ.

Lemma 2.1 gives us a spectral sequence, whose E^2 term is

$$\varprojlim{}^i \operatorname{Ext}^j_R(M_\lambda, N),$$

which converges to $\operatorname{Ext}^{i+j}_R(M,N)$. Once again, we have that

$$\varprojlim{}^{n+2} \operatorname{Ext}^0_R(M_\lambda, N) = \operatorname{PExt}^{n+2}(M,N),$$

and this does not vanish, by the choice of M. But $\operatorname{Ext}^{n+2}_R(M,N) = 0$, since N is of injective dimension at most n, so there must be a non-zero differential into the term

$$\varprojlim{}^{n+2} \operatorname{Ext}^0_R(M_\lambda, N).$$

Now observe that

$$\varprojlim{}^0 \operatorname{Ext}^{n+1}_R(M_\lambda, N) = 0,$$

since N is of injective dimension at most n. It follows that for some i with $1 \leq i \leq n$, there is a non-zero differential in the spectral sequence, from

$$\varprojlim{}^i \operatorname{Ext}^{n+1-i}_R(M_\lambda, N)$$

to the term $\varprojlim{}^{n+2} \operatorname{Ext}^0_R(M_\lambda, N) \neq 0$.

Now recall the construction of our spectral sequence, from Lemma 2.1. Since M is the filtered colimit of M_λ, there is an exact resolution of M

$$\cdots \longrightarrow \bigoplus_{\lambda \to \mu} M_\lambda \longrightarrow \bigoplus_{\lambda} M_\lambda \longrightarrow M \longrightarrow 0.$$

This resolution is a pure exact resolution by pure projectives. (It is pure exact because it remains exact in the category $\operatorname{Mod}(\text{mod-}R)$. And direct sums of finitely presented modules M_λ are pure projective.) By Lemma 1.3, it becomes an exact resolution by projectives in the category $\text{Mod-}\mathcal{T}^c$.

To simplify the notation, we will write the above resolution as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Let K_i stand for the image of the map $P_i \longrightarrow P_{i-1}$. In Lemma 2.4 we showed that

$$\varprojlim{}^i \operatorname{Ext}^{n+1-i}_R(M_\lambda, N)$$

is the group of extensions

$$\operatorname{Ext}^i(\mathbf{y}M, \mathbf{y}\Sigma^{n+1-i}N).$$

But since the pure exact sequence

$$0 \longrightarrow K_{i-1} \longrightarrow P_{i-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

remains exact in Mod-$\mathcal{T}^c$, and the middle modules map to projectives in Mod-$\mathcal{T}^c$, we deduce that the above extension group is isomorphic to

$$\mathrm{Ext}^1(\mathbf{y}K_{i-1}, \mathbf{y}\Sigma^{n+1-i}N).$$

In other words, an element of the group

$$\varprojlim{}^i \mathrm{Ext}_R^{n+1-i}(M_\lambda, N)$$

may be thought of as a short exact sequence in Mod-$\mathcal{T}^c$

$$0 \longrightarrow \mathbf{y}\Sigma^{n+1-i}N \longrightarrow F \longrightarrow \mathbf{y}K_{i-1} \longrightarrow 0.$$

We know that in the spectral sequence, for some $1 \leq i \leq n$, there is a non-zero differential

$$\varprojlim{}^i \mathrm{Ext}_R^{n+1-i}(M_\lambda, N) \quad \supset \quad E \quad \xrightarrow{\gamma} \quad \varprojlim{}^{n+2} \mathrm{Ext}_R^0(M_\lambda, N),$$

for a subgroup $E \subset \varprojlim{}^i \mathrm{Ext}_R^{n+1-i}(M_\lambda, N)$. What we will now show is that, if $\gamma(x) \neq 0$, then x corresponds to an exact sequence

$$0 \longrightarrow \mathbf{y}\Sigma^{n+1-i}N \longrightarrow F \longrightarrow \mathbf{y}K_{i-1} \longrightarrow 0$$

where F is not isomorphic to any $\mathbf{y}Y$. Expressing the same thing slightly differently, we will show that if $x \in \varprojlim{}^i \mathrm{Ext}_R^{n+1-i}(M_\lambda, N)$ comes from an exact sequence of functors with $F = \mathbf{y}Y$, then $\gamma(x) = 0$.

Suppose therefore that we are given a short exact sequence in Mod-$\mathcal{T}^c$

$$0 \longrightarrow \mathbf{y}\Sigma^{n+1-i}N \longrightarrow \mathbf{y}Y \longrightarrow \mathbf{y}K_{i-1} \longrightarrow 0.$$

We need to show that in the spectral sequence, the differential γ annihilates x. By Lemma 2.8, the exact sequence of functors comes from a triangle

$$\Sigma^{n+1-i}N \longrightarrow Y \longrightarrow K_{i-1} \xrightarrow{\partial} \Sigma^{n+2-i}N$$

with ∂ a phantom map. From the definition of the modules K_i, we have a pure exact sequence of R–modules

$$0 \longrightarrow K_i \longrightarrow P_{i-1} \longrightarrow K_{i-1} \longrightarrow 0.$$

This exact sequence gives a triangle in $\mathcal{T} = D(R)$. The fact that $\partial : K_{i-1} \longrightarrow \Sigma^{n+2-i}N$ is phantom tells us that the composite

$$P_{i-1} \longrightarrow K_{i-1} \xrightarrow{\partial} \Sigma^{n+2-i}N$$

must vanish, since P_{i-1} is a coproduct of compact objects. But then the map ∂ must factor as

$$K_{i-1} \longrightarrow \Sigma K_i \longrightarrow \Sigma^{n+2-i}N.$$

Thus if an element $x \in \varprojlim{}^i \mathrm{Ext}_R^{n+1-i}(M_\lambda, N)$ comes from a short exact sequence

$$0 \longrightarrow \mathbf{y}\Sigma^{n+1-i}N \longrightarrow F \longrightarrow \mathbf{y}K_{i-1} \longrightarrow 0$$

with $F \simeq \mathbf{y}Y$, then Y is determined by a class

$$y \in \mathrm{Ext}_R^{n+1-i}(K_i, N).$$

In conclusion, we deduce the following. Let us define $K_0 = 0$. We have a map of chain complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_i & \longrightarrow & P_{i-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{0} & K_i & \xrightarrow{0} & K_{i-1} & \xrightarrow{0} & \cdots & \xrightarrow{0} & K_1 & \xrightarrow{0} & K_0 & \longrightarrow & 0\ .
\end{array}$$

Hence there is a map of spectral sequences in hypercohomology. On the E^2 term, it is

$$\mathrm{Ext}_R^j(K_i, N) \longrightarrow \varprojlim{}^i \mathrm{Ext}^j(M_\lambda, N).$$

The whole point is that the spectral sequence on the left degenerates at E^1, since it comes from a complex with zero differentials. We have shown that if $x \in \varprojlim{}^i \mathrm{Ext}^{n+1-i}(M_\lambda, N)$ corresponds to an extension

$$0 \longrightarrow \mathbf{y}\Sigma^{n+1-i}N \longrightarrow F \longrightarrow \mathbf{y}K_{i-1} \longrightarrow 0$$

with $F \simeq \mathbf{y}Y$, then x is the image of some y from the trivial spectral sequence. Therefore, all differentials out of x vanish. □

Example 2.12. Let k be an infinite field of cardinality $\aleph_t$. Then by [4], the polynomial ring $R = k[x, y]$ is of pure global dimension $t + 1$ (∞ if t is infinite). On the other hand, it is of global dimension 2. Hence there do exist modules N over $R = k[x, y]$, satisfying the assumptions of the theorem when t is at least 3.

We can give a refinement of our results for when the ring R is hereditary; recall that R is *hereditary* if its global dimension is ≤ 1. Examples of hereditary rings are commutative principal ideal domains, and non-commutative polynomial rings.

Theorem 2.13. *Let R be a hereditary ring. Then*

(i) *[BRM] holds in $\mathcal{T}$ if and only if the pure global dimension of R is at most* 1*; and*
(ii) *[BRO] holds in $\mathcal{T}$ if and only if the pure global dimension of R is at most* 2.

Proof. (i) holds by Neeman's theorem 0.7, combined with the equality we prove in Proposition 1.4: for hereditary rings

$$\mathrm{pgldim}\ R = \mathrm{pgldim}\ D(R).$$

For (ii), note that Beligiannis' result (Theorem 0.9) tells us, that [BRO] holds if $\mathrm{pgldim}\ D(R) \leq 2$. The converse comes from Proposition 2.9 which says that if N is an R–module of injective dimension ≤ 1 and $\mathrm{PExt}^3(M, N) \neq 0$, then [BRO] fails for $\mathcal{T} = D(R)$. Thus if R is hereditary but of pure global dimension ≥ 3, [BRO] must fail. (Here we have used the easy fact that every hereditary ring is coherent.) □

Let k be a field. In our counterexamples, we always consider k-linear triangulated categories $\mathcal{T}$. When $\mathcal{T}$ is k-linear, an additive functor $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ always extends uniquely to a k-linear functor $\{\mathcal{T}^c\}^{op} \longrightarrow \text{Mod-}k$, so we can restrict attention to such k-linear functors. The following lemma shows that our counterexamples must take values in infinite-dimensional vector spaces. The idea of the double dual used in the proof is due to M. Van den Bergh.

Lemma 2.14. *Let k be a field and*

$$F : \{\mathcal{T}^c\}^{op} \longrightarrow \text{mod-}k$$

an exact functor which takes its values in the category mod-k *of finite-dimensional vector spaces. Then F is of the form $\mathbf{y}X$ for some $X \in \mathcal{T}$.*

Proof. Denote by D the functor which takes a vector space to its dual. Then the functor $G = D \circ F$ is exact and covariant. Let

$$\tilde{G} : \mathcal{T} \longrightarrow \text{Mod-}k$$

be the Kan extension of G to $\mathcal{T}$. Thus, for $Y \in \mathcal{T}$, we have

$$\tilde{G}(Y) = \operatorname*{colim}_{\longrightarrow} GC,$$

where the colimit is taken over the category of arrows $C \longrightarrow Y$ from a compact C to Y. A moment's thought will convince the reader that $\tilde{G}$ is exact and commutes with coproducts (cf. Prop. 2.3 of [28]). Hence $D \circ \tilde{G}$ is exact and takes coproducts to products. By Brown's theorem, it is representable: We have

$$D \circ \tilde{G} = \mathcal{T}(-, X)$$

for some $X \in \mathcal{T}$. We claim that $\mathbf{y}X = F$. Indeed, the restriction of $D \circ \tilde{G}$ to $\mathcal{T}^c$ is isomorphic to $D \circ D \circ F$, and this functor is isomorphic to F because FC is finite-dimensional for all $C \in \mathcal{T}^c$. □

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