QUILLEN MODEL STRUCTURES FOR RELATIVE HOMOLOGICAL ALGEBRA

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Abstract. An important example of a model category is the category of unbounded chain complexes of R-modules, which has as its homotopy category the derived category of the ring R. This example shows that traditional homological algebra is encompassed by Quillen's homotopical algebra. The goal of this paper is to show that more general forms of homological algebra also fit into Quillen's framework. Specifically, a projective class on a complete and cocomplete abelian category $\mathcal{A}$ is exactly the information needed to do homological algebra in $\mathcal{A}$. The main result is that, under weak hypotheses, the category of chain complexes of objects of $\mathcal{A}$ has a model category structure that reflects the homological algebra of the projective class in the sense that it encodes the Ext groups and more general derived functors. Examples include the "pure derived category" of a ring R, and derived categories capturing relative situations, including the projective class for Hochschild homology and cohomology. We characterize the model structures that are cofibrantly generated, and show that this fails for many interesting examples. Finally, we explain how the category of simplicial objects in a possibly non-abelian category can be equipped with a model category structure reflecting a given projective class, and give examples that include equivariant homotopy theory and bounded below derived categories.

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Introduction

An important example of a model category is the category $\mathrm{Ch}(R)$ of unbounded chain complexes of R-modules, which has as its homotopy category the derived category $\mathcal{D}(R)$ of the associative ring R. The formation of a projective resolution is an example of cofibrant replacement, traditional derived functors are examples of derived functors in the model category sense, and Ext groups appear as groups of maps in the derived category. This example shows that traditional homological algebra is encompassed by Quillen's homotopical algebra, and indeed this unification was one of the main points of Quillen's influential work [Qui67].

The goal of this paper is to illustrate that more general forms of homological algebra also fit into Quillen's framework. In any abelian category $\mathcal{A}$ there is a natural notion of "projective object" and "epimorphism." However, it is sometimes useful to impose different definitions of these terms. If this is done in a way that satisfies some natural axioms, what is obtained is a "projective class," which is exactly the information needed to do homological algebra in $\mathcal{A}$. Our main result shows that for a wide variety of projective classes (including all those that arise in examples) the category of unbounded chain complexes of objects of $\mathcal{A}$ has a model category structure that reflects the homological algebra of the projective class in the same way that ordinary homological algebra is captured by the usual model structure on $\mathrm{Ch}(R)$.

When $\mathcal{A}$ has enough projectives, the projective objects and epimorphisms form a projective class. Therefore the results of this paper apply to traditional homological algebra as well. Even in this special case, it is not a trivial fact that the category of *unbounded* chain complexes can be given a model category structure, and indeed Quillen restricted himself to the bounded below case. We know of three other written proofs that the category of unbounded chain complexes is a model category [Gro, Hin97, Hov98], which do the case of R-modules, but this was probably known to others as well.

An important corollary of the fact that a derived category $\mathcal{D}(\mathcal{A})$ is the homotopy category of a model category is that the group $\mathcal{D}(\mathcal{A})(X, Y)$ of maps is a set (as opposed to a proper class) for any two chain complexes X and Y. This is not the case in general, and much work on derived categories ignores this possibility. The importance of this point is that if one uses the morphisms in the derived category to index constructions in other categories or to define cohomology groups, one needs to know that the indexing class is actually a set. Recently, $\mathcal{D}(\mathcal{A})(X, Y)$ has been shown to be a set under various assumptions on $\mathcal{A}$. (See Weibel [Wei94] Remark 10.4.5, which credits Gabber, and Exercise 10.4.5, which credits Lewis, May and Steinberger [LMS86]. See also Kriz and May [KM95, Part III].) The assumptions that appear in the present paper are different from those that have appeared before and the proof is somewhat easier (because of our use of the theory of cofibrantly generated model categories), so this paper may be of some interest even in this special case.

Another consequence of the fact that $\mathrm{Ch}(R)$ is a model category is the existence of resolutions coming from cofibrant and fibrant approximations, and the related derived functors. Some of these are discussed in [AFH97] and [Spa88]. We do not discuss these topics here, but just mention that these resolutions are immediate once you have the model structure, so our approach gives these results with very little work.

While our results include new examples of traditional homological algebra, our focus is on more general projective classes. For example, let A be an algebra over a commutative ring k. We call a map of A-bimodules a relative epimorphism if it is split epic as a map of k-modules, and we call an A-bimodule a relative projective if maps from it lift over relative epimorphisms. These definitions give a projective class, and Theorem 2.2 tells us that there is a model category, and therefore a derived category, that captures the homological algebra of this situation. For example, Hochschild cohomology groups appear as Hom sets in this derived category (see Example 3.7).

We also discuss pure homological algebra and construct the "pure derived category" of a ring. Pure homological algebra has applications to phantom maps in the stable homotopy category [CS98] and in the (usual) derived category of a ring [Chr98], connections to Kasparov KK-theory [Sch01], and is actively studied by algebraists and representation theorists.

In the last section we describe a model category structure on the category of non-negatively graded chain complexes that works for an arbitrary projective class on an abelian category, without any hypotheses. More generally, we show that under appropriate hypotheses a projective class on a possibly non-abelian category $\mathcal{A}$ determines a model category structure on the category of simplicial objects in $\mathcal{A}$. As an example, we deduce that the category of equivariant simplicial sets has various model category structures.

We now briefly outline the paper. In Section 1 we give the axioms for a projective class and mention many examples that will be discussed further in Subsections 3.1 and 5.3. In Section 2 we describe the desired model structure coming from a projective class and state our main theorem, which says that the model structure exists as long as cofibrant replacements exist. We also give two hypotheses that each imply the existence of cofibrant replacements. The first hypothesis handles situations coming from adjoint pairs, and is proved to be sufficient in Section 3, where we also give many examples involving relative situations. The second hypothesis deals with projective classes that have enough small projectives and is proved to be sufficient in Section 4. In Section 5 we prove that the model structure that one gets is cofibrantly generated if and only if there is a *set* of enough small projectives. We do this using the recognition lemma for cofibrantly generated categories, which is recalled in Subsection 5.1. This case is proved from scratch, independent of the main result in Section 2, since the proof is not long. In Subsection 5.3 we give two examples, the traditional derived category of R-modules and the pure derived category. We describe how the two relate and why the pure derived category is interesting. In the final section we discuss the bounded below case, which works for *any* projective class, and describe a result for simplicial objects in a possibly non-abelian category.

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0. Notation and conventions

We make a few blanket assumptions. With the exception of Section 6, $\mathcal{A}$ and $\mathcal{B}$ will denote abelian categories. We will assume that our abelian categories are bicomplete; this assumption is stronger than strictly necessary, but it simplifies the statements of our results. For any category $\mathcal{C}$, we write $\mathcal{C}(A, B)$ for the set of maps from A to B in $\mathcal{C}$.

We write $\mathrm{Ch}(\mathcal{A})$ for the category of unbounded chain complexes of objects of $\mathcal{A}$ and degree zero chain maps. To fix notation, assume that the differentials lower degree. For an object X of $\mathrm{Ch}(\mathcal{A})$, define $Z_nX := \ker(d\colon X_n \to X_{n-1})$ and $B_nX := \mathrm{im}(d\colon X_{n+1} \to X_n)$, and write H_nX for the quotient. A map inducing an isomorphism in H_n for all n is a **quasi-isomorphism**. The **suspension** ΣX of X has $(\Sigma X)_n = X_{n-1}$ and $d_{\Sigma X} = -d_X$. The functor Σ is defined on morphisms by $(\Sigma f)_n = f_{n-1}$. Given a map $f\colon X \to Y$ of chain complexes, the **cofibre of f** is the chain complex C with $C_n = Y_n \oplus X_{n-1}$ and with differential $d(y, x) = (dy + fx, -dx)$. There are natural maps $Y \to C \to \Sigma X$, and the sequence $X \to Y \to C \to \Sigma X$ is called a **cofibre sequence**.

Two maps $f, g\colon X \to Y$ are **chain homotopic** if there is a collection of maps $s_n\colon X_n \to Y_{n+1}$ such that $f - g = ds + sd$. We write $[X, Y]$ for the chain homotopy classes of maps from X to Y and $\mathcal{K}(\mathcal{A})$ for the category of chain complexes and chain homotopy classes of maps. Two complexes are **chain homotopy equivalent** if they are isomorphic in $\mathcal{K}(\mathcal{A})$, and a complex is **contractible** if it is chain homotopy equivalent to 0. $\mathcal{K}(\mathcal{A})$ is a triangulated category, with triangles the sequences chain homotopy equivalent to the cofibre sequences above. The functors $H_n(-)$, $[X,-]$ and $[-,Y]$ are defined on $\mathcal{K}(\mathcal{A})$ and send triangles to long exact sequences.

For P in $\mathcal{A}$, D^kP denotes the (contractible) complex such that $(D^kP)_n = P$ if $n = k$ or $n = k-1$ but $(D^kP)_n = 0$ for other values of n, and whose differential is the identity in degree k. The functor D^k is left adjoint to the functor $X \mapsto X_k$, and right adjoint to the functor $X \mapsto X_{k-1}$. The path complex PX of a complex X is the contractible complex such that $(PX)_n = X_n \oplus X_{n+1}$, where $d(x,y) = (dx, x - dy)$.

We assume knowledge of the basics of model categories, for which [DS95] is an excellent reference. We use the definition of model category that requires that the category be complete and cocomplete, and that the factorizations be functorial. Correspondingly, when we say that cofibrant replacements exist, we implicitly mean that they are functorial.

1. Projective classes

1.1. **Definition and some examples.** In this subsection we explain the notion of a projective class, which is the information necessary in order to do homological algebra. Intuitively, a projective class is a choice of which sort of "elements" we wish to think about. In this section we focus on the case of an abelian category, but this definition works for any pointed category with kernels.

The elements of a set X correspond bijectively to the maps from a singleton to X, and the elements of an abelian group A correspond bijectively to the maps from $\mathbb{Z}$ to A. Motivated by this, we call a map $P \to A$ in any category a ***P*-element of *A***. If we don't wish to mention P, we call such a map a **generalized element of *A***. A map $A \to B$ in any category is determined by what it does on generalized elements. If $\mathcal{P}$ is a collection of objects, then a **$\mathcal{P}$-element** means a P-element for some P in $\mathcal{P}$.

Let $\mathcal{A}$ be an abelian category. A map $B \to C$ is said to be ***P*-epic** if it induces a surjection of P-elements, that is, if the induced map $\mathcal{A}(P,B) \to \mathcal{A}(P,C)$ is a surjection of abelian groups. The map $B \to C$ is **$\mathcal{P}$-epic** if it is P-epic for all P in $\mathcal{P}$.

Definition 1.1. A **projective class** on $\mathcal{A}$ is a collection $\mathcal{P}$ of objects of $\mathcal{A}$ and a collection $\mathcal{E}$ of maps in $\mathcal{A}$ such that

(*i*) $\mathcal{E}$ is precisely the collection of all $\mathcal{P}$-epic maps;
(*ii*) $\mathcal{P}$ is precisely the collection of all objects P such that each map in $\mathcal{E}$ is P-epic;
(*iii*) for each object B there is a map $P \to B$ in $\mathcal{E}$ with P in $\mathcal{P}$.

When a collection $\mathcal{P}$ is part of a projective class $(\mathcal{P},\mathcal{E})$, the projective class is unique, and so we say that $\mathcal{P}$ determines a projective class or even that $\mathcal{P}$ is a projective class. An object of $\mathcal{P}$ is called a **$\mathcal{P}$-projective**, or, if the context is clear, a **relative projective**.

A sequence
$$A \to B \to C$$
is said to be ***P*-exact** if the composite $A \to C$ is zero and
$$\mathcal{A}(P,A) \to \mathcal{A}(P,B) \to \mathcal{A}(P,C)$$
is an exact sequence of abelian groups. The latter can be rephrased as the condition that $A \to B \to C$ induces an exact sequence of P-elements. A **$\mathcal{P}$-exact sequence** is one that is P-exact for all P in $\mathcal{P}$.

Example 1.2. For an associative ring R, let $\mathcal{A}$ be the category of left R-modules, let $\mathcal{P}$ be the collection of all summands of free R-modules and let $\mathcal{E}$ be the collection of all surjections of R-modules. Then $\mathcal{E}$ is precisely the collection of $\mathcal{P}$-epimorphisms, and $\mathcal{P}$ is a projective class. The $\mathcal{P}$-exact sequences are the usual exact sequences.

Example 1.2 is a **categorical** projective class in the sense that the $\mathcal{P}$-epimorphisms are just the epimorphisms and the $\mathcal{P}$-projectives are the categorical projectives, *i.e.,* those objects P such that maps from P lift through epimorphisms.

Here are two examples of non-categorical projective classes.

Example 1.3. If $\mathcal{A}$ is any abelian category, $\mathcal{P}$ is the collection of all objects, and $\mathcal{E}$ is the collection of all split epimorphisms $B \to C$, then $\mathcal{P}$ is a projective class. It is called the **trivial projective class**. A sequence $A \to B \to C$ is $\mathcal{P}$-exact if and only if $A \to \ker(B \to C)$ is split epic.

Example 1.4. Let $\mathcal{A}$ be the category of left R-modules, as in Example 1.2. Let $\mathcal{P}$ consist of all summands of sums of finitely presented modules and define $\mathcal{E}$ to consist of all $\mathcal{P}$-epimorphisms. Then $\mathcal{P}$ is a projective class. A sequence is $\mathcal{P}$-exact iff it is exact after tensoring with every right module.

Examples 1.2 and 1.4 will be discussed further in Subsection 5.3. Example 1.3 is important because many interesting examples are "pullbacks" of this projective class (see Subsection 1.3).

Let $\mathcal{P}$ be a projective class. If $\mathcal{S}$ is a subcollection of $\mathcal{P}$ (not necessarily a set), and if a map is $\mathcal{S}$-epic iff it is $\mathcal{P}$-epic, then we say that $\mathcal{P}$ is **determined by $\mathcal{S}$** and that $\mathcal{S}$ is a collection of **enough projectives**. Some projective classes, such as Examples 1.2 and 1.4, are determined by a set, and the lemma below shows that any set of objects determines a projective class. The trivial projective class is sometimes not determined by a set (see Subsection 5.4).

Lemma 1.5. *Suppose $\mathcal{F}$ is any* set *of objects in an abelian category with coproducts. Let $\mathcal{E}$ be the collection of $\mathcal{F}$-epimorphisms and let $\mathcal{P}$ be the collection of all objects P such that every map in $\mathcal{E}$ is P-epic. Then $\mathcal{P}$ is the collection of retracts of coproducts of objects of $\mathcal{F}$ and $(\mathcal{P}, \mathcal{E})$ is a projective class.*

Proof. Given an object X, let P be the coproduct $\coprod F$ indexed by all maps $F \to X$ and all objects F in $\mathcal{F}$. The natural map $P \to X$ is clearly an $\mathcal{F}$-epimorphism. Moreover, if X is in $\mathcal{P}$, then this map is split epic, and so X is a retract of a coproduct of objects of $\mathcal{F}$. These two facts show that $(\mathcal{P}, \mathcal{E})$ is a projective class. $\square$

1.2. **Homological algebra.** A projective class is precisely the information needed to form projective resolutions and define derived functors. All of the usual definitions and theorems go through. A **$\mathcal{P}$-resolution** of an object M is a $\mathcal{P}$-exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

such that each P_i is in $\mathcal{P}$. If $\mathcal{B}$ is an abelian category and $T\colon \mathcal{A} \to \mathcal{B}$ is an additive functor, then the nth left derived functor of T with respect to $\mathcal{P}$ is defined by $L_n^{\mathcal{P}} T(M) = H_n(T(P_*))$ where P_* is a $\mathcal{P}$-resolution of M. One has the usual uniqueness of resolutions up to chain homotopy and so this is well-defined. From a $\mathcal{P}$-exact sequence $0 \to L \to M \to N \to 0$ one gets a long exact sequence involving the derived functors. The abelian groups $\operatorname{Ext}^n_{\mathcal{P}}(M, N)$ can be defined in the usual two ways, as equivalence classes of $\mathcal{P}$-exact sequences $0 \to N \to L_1 \to \cdots \to L_n \to M \to 0$, or as $L_n^{\mathcal{P}} T(M)$ where $T(-) = \mathcal{A}(-, N)$.

For further details and useful results we refer the reader to the classic reference [EM65].

1.3. **Pullbacks.** A common setup in relative homological algebra is the following. We assume we have a functor $U\colon \mathcal{A} \to \mathcal{B}$ of abelian categories, together with a left adjoint $F\colon \mathcal{B} \to \mathcal{A}$. Then U and F are additive, U is left exact and F is right exact.

If $(\mathcal{P}', \mathcal{E}')$ is a projective class on $\mathcal{B}$, we define $\mathcal{P} := \{\text{retracts of } FP \text{ for } P \text{ in } \mathcal{P}'\}$ and $\mathcal{E} := \{B \to C \text{ such that } UB \to UC \text{ is in } \mathcal{E}'\}$. Then one can easily show that $(\mathcal{P}, \mathcal{E})$ is a projective class on $\mathcal{A}$ and that a sequence is $\mathcal{P}$-exact if and only if it is sent to a $\mathcal{P}'$-exact sequence by U. $(\mathcal{P}, \mathcal{E})$ is called the **pullback** of $(\mathcal{P}', \mathcal{E}')$ along the right adjoint U.

The most common case is when $(\mathcal{P}', \mathcal{E}')$ is the trivial projective class (see Example 1.3). Then for any M in $\mathcal{A}$ the counit $FUM \to M$ is a $\mathcal{P}$-epimorphism from a $\mathcal{P}$-projective.

Example 1.6. Let $R \to S$ be a map of associative rings. Write R-mod and S-mod for the categories of left R- and S-modules. Consider the forgetful functor $U\colon S\text{-mod} \to R\text{-mod}$ and its left adjoint F that sends an R-module M to $S \otimes_R M$. The pullback along U of the trivial projective class on R-mod gives a projective class $\mathcal{P}$. The $\mathcal{P}$-projectives are the S-modules P such that the natural map $S \otimes_R P \to P$ is split epic as a map of S-modules. The $\mathcal{P}$-epimorphisms are the S-module maps that are split epic as maps of R-modules.

Example 1.7. As above, let $R \to S$ be a map of rings. The forgetful functor $U\colon S\text{-mod} \to R\text{-mod}$ has a right adjoint G that sends an R-module M to the S-module $R\text{-mod}(S, M)$. We can pullback the trivial injective class along U to get an injective class on S-mod. (An injective class is just a projective class on the opposite category.) The relative injectives are the S-modules I such that the natural map $I \to R\text{-mod}(S, I)$ is split monic as a map of S-modules, and the relative monomorphisms are the S-module maps that are split monic as maps of R-modules.

We investigate these examples in detail in Section 3.

1.4. **Strong projective classes.** In Section 3 we will focus on projective classes that are the pullback of a trivial projective class along a right adjoint. In this subsection we describe a special property that these projective classes have.

Definition 1.8. A projective class $\mathcal{P}$ is **strong** if for each $\mathcal{P}$-projective P and each $\mathcal{P}$-epimorphism $M \to N$, the surjection $\mathcal{A}(P, M) \to \mathcal{A}(P, N)$ of abelian groups is *split* epic.

It is clear that a trivial projective class is strong, and that the pullback of a strong projective class is strong. The importance of strong projective classes comes from the following lemma.

Lemma 1.9. *The following are equivalent for a projective class $\mathcal{P}$ on an abelian category $\mathcal{A}$:*

(*i*) *$\mathcal{P}$ is strong.*
(*ii*) *For each complex C in $Ch(\mathcal{A})$, if the complex $\mathcal{A}(P, C)$ in $Ch(\mathcal{A}b)$ has trivial homology for each P in $\mathcal{P}$, then it is contractible for each P in $\mathcal{P}$.*
(*iii*) *For each map f in $Ch(\mathcal{A})$, if the map $\mathcal{A}(P, f)$ in $Ch(\mathcal{A}b)$ is a quasi-isomorphism for each P in $\mathcal{P}$, then it is a chain homotopy equivalence for each P in $\mathcal{P}$.*

Here $\mathcal{A}(P, C)$ denotes the chain complex with $\mathcal{A}(P, C_k)$ in degree k. We also use the notation from Section 0.

Proof. (i) $\implies$ (ii): Assume $\mathcal{P}$ is a strong projective class and let C be a complex in $\mathrm{Ch}(\mathcal{A})$ such that $\mathcal{A}(P, C)$ has trivial homology for each $\mathcal{P}$-projective P. Then for each k and each P we have a short exact sequence

$$0 \to \mathcal{A}(P, Z_k C) \to \mathcal{A}(P, C_k) \to \mathcal{A}(P, Z_{k-1} C) \to 0$$

of abelian groups. Because the projective class is strong, the sequence is split. This implies that the complex $\mathcal{A}(P, C)$ is isomorphic to $\oplus_k D^{k+1} \mathcal{A}(P, Z_k C)$, and in particular that it is contractible.

(ii) $\implies$ (iii): Let $f\colon X \to Y$ be a map in $\mathrm{Ch}(\mathcal{A})$ such that $\mathcal{A}(P, X) \to \mathcal{A}(P, Y)$ is a quasi-isomorphism for each P in $\mathcal{P}$, and let C be the cofibre of f. Then by the long exact sequence, $\mathcal{A}(P, C)$ has trivial homology for each P in $\mathcal{P}$. By (ii) this complex is contractible. This implies that $\mathcal{A}(P, X) \to \mathcal{A}(P, Y)$ is a chain homotopy equivalence.

(iii) $\implies$ (i): Let $M \to N$ be a $\mathcal{P}$-epimorphism with kernel L. Then the complex $L \to M \to N$ has trivial homology after applying $\mathcal{A}(P, -)$, for each $\mathcal{P}$-projective P. By (iii), it is contractible after applying $\mathcal{A}(P,-)$, for each P. In particular, $\mathcal{A}(P,M) \to \mathcal{A}(P,N)$ is split epic. □

2. The relative model structure

The object of this section is to construct a Quillen model structure on the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$ that reflects a given projective class $\mathcal{P}$ on $\mathcal{A}$.

If X is a chain complex, we write $\mathcal{A}(P,X)$ for the chain complex that has the abelian group $\mathcal{A}(P,X_n)$ in degree n. This is the chain complex of P-elements of X.

Definition 2.1. A map $f\colon X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ is a **$\boldsymbol{\mathcal{P}}$-equivalence** if $\mathcal{A}(P,f)$ is a quasi-isomorphism in $\mathrm{Ch}(\mathbb{Z})$ for each P in $\mathcal{P}$. The map f is a **$\boldsymbol{\mathcal{P}}$-fibration** if $\mathcal{A}(P,f)$ is a surjection for each P in $\mathcal{P}$. The map f is a **$\boldsymbol{\mathcal{P}}$-cofibration** if f has the left lifting property with respect to all maps that are both $\mathcal{P}$-fibrations and $\mathcal{P}$-equivalences (the **$\boldsymbol{\mathcal{P}}$-trivial fibrations**).

The motivation for this definition is that it implies that a complex

$$\cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots$$

equipped with an augmentation $P_0 \to M$ to an object M is a cofibrant replacement if and only if it is a $\mathcal{P}$-resolution in the sense of Subsection 1.2. This implies that if M and N are objects of $\mathcal{A}$ thought of as complexes concentrated in degree zero, then $\mathrm{Ext}^n_{\mathcal{P}}(M,N)$ can be identified with maps from $\Sigma^n M$ to N in the homotopy category of the model category $\mathrm{Ch}(\mathcal{A})$. This will be described in more detail in Subsection 2.1.

The main goal of this section is then to prove the following theorem.

Theorem 2.2. *Suppose $\mathcal{P}$ is a projective class on the abelian category $\mathcal{A}$. Then the category Ch($\mathcal{A}$), together with the $\mathcal{P}$-equivalences, the $\mathcal{P}$-fibrations, and the $\mathcal{P}$-cofibrations, forms a Quillen model category if and only if cofibrant replacements exist. When the model structure exists, it is proper. Cofibrant replacements exist in each of the following cases:*

- A: *$\mathcal{P}$ is the pullback of the trivial projective class along a right adjoint that preserves countable sums.*
- B: *There are enough κ-small $\mathcal{P}$-projectives for some cardinal κ, and $\mathcal{P}$-resolutions can be chosen functorially.*

The words "enough" and "κ-small" will be explained in Section 4.

We call this structure the **relative model structure**. We point out that we are using the modern definition of model category [DHK97, Hov98], so our factorizations will be functorial. Correspondingly, we require our cofibrant replacements to be functorial as well. This theorem requires our blanket assumption that abelian categories are bicomplete.

In Subsection 2.1 we will describe further properties of these model structures, including conditions under which they are monoidal. In Section 5 we show that if there is a *set* of enough small projectives, then the model structure is cofibrantly generated. On the other hand, we show in Subsection 5.4 that model categories coming from Case A are generally not cofibrantly generated.

In Subsection 2.2 we explain why it is no loss of generality to start with a projective class $(\mathcal{P},\mathcal{E})$, rather than just an arbitrary class $\mathcal{P}$ of test objects.

Proof. Some of the properties necessary for a model category are evident from the definitions. It is clear that $\mathrm{Ch}(\mathcal{A})$ is bicomplete, since $\mathcal{A}$ is so. Also, $\mathcal{P}$-equivalences have the two out of three property, and $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations, and $\mathcal{P}$-cofibrations are closed under retracts. Furthermore, $\mathcal{P}$-cofibrations have the left lifting property with respect to $\mathcal{P}$-trivial fibrations, by definition. It remains to show that $\mathcal{P}$-trivial cofibrations have the left lifting property with respect to $\mathcal{P}$-fibrations, and that the two factorization axioms hold. The remaining lifting property will be proved in Proposition 2.8, and the two factorization axioms will be proved in Propositions 2.9 and 2.10, assuming that cofibrant replacements exist. Properness will be proved in Proposition 2.18, which also defines the term.

That cofibrant replacements exist in cases A and B will be proved in Sections 3 and 4, respectively. □

Note that our Theorem 2.2 will also apply when we have an injective class, that is, a projective class on $\mathcal{A}^{\mathrm{op}}$, by dualizing the definition of the model structure.

We continue the proof of Theorem 2.2 with a lemma that gives us a simple test of the lifting property. We use the notation from Section 0.

Lemma 2.3. *Suppose $p\colon X \to Y$ is a $\mathcal{P}$-fibration with kernel K, and $i\colon A \to B$ is a degreewise split inclusion whose cokernel C is a complex of relative projectives. If every map $C \to \Sigma K$ is chain homotopic to 0, then i has the left lifting property with respect to p.*

Proof. We can write $B_n \cong A_n \oplus C_n$, where the differential is defined by $d(a,c) = (da+\tau c, dc)$ (we use the element notation for convenience, but it is not strictly necessary), and $\tau\colon C_n \to A_{n-1}$ can be any family of maps such that $d\tau + \tau d = 0$. Suppose we have a commutative square as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

In terms of the splitting $B_n \cong A_n \oplus C_n$, we have $g(a,c) = pf(a) + \alpha(c)$, where the family $\alpha_n\colon C_n \to Y_n$ satisfies $d\alpha = pf\tau + \alpha d$. We are looking for a map $h\colon B \to X$ making the diagram above commute. In terms of the splitting, this means we are looking for a family of maps $\beta_n\colon C_n \to X_n$ such that $p\beta = \alpha$ and $d\beta = f\tau + \beta d$. Since C_n is relatively projective, and p is $\mathcal{P}$-epic, there is a map $\gamma\colon C_n \to X_n$ such that $p\gamma = \alpha$. The difference $\delta = d\gamma - f\tau - \gamma d$ may not be zero, but at least $p\delta = 0$. Let $j\colon K \to X$ denote the kernel of p. Then there is a map $F\colon C_n \to K_{n-1}$ such that $jF = \delta$. Furthermore, one can check that $Fd = -dF$, so that $F\colon C \to \Sigma K$ is a chain map. By hypothesis, F is chain homotopic to 0 by a map $D\colon C_n \to K_n$, so that $Dd - dD = F$. Define $\beta = \gamma + jD$. Then β defines the desired lift, so i has the left lifting property with respect to p. □

Now we study the $\mathcal{P}$-cofibrations. A complex C is called **$\mathcal{P}$-cofibrant** if the map $0 \to C$ is a $\mathcal{P}$-cofibration. A complex K is called **weakly $\mathcal{P}$-contractible** if the map $K \to 0$ is a $\mathcal{P}$-equivalence, or, equivalently, if all maps from a complex $\Sigma^k P$ consisting of a relative projective concentrated in one degree to K are chain homotopic to 0.

Lemma 2.4. *A complex C is $\mathcal{P}$-cofibrant if and only if each C_n is relatively projective and every map from C to a weakly $\mathcal{P}$-contractible complex K is chain homotopic to 0.*

Proof. Suppose first that C is $\mathcal{P}$-cofibrant. If $M \to N$ is a $\mathcal{P}$-epimorphism, then the map $D^{n+1}M \to D^{n+1}N$ is a $\mathcal{P}$-fibration. It is also a $\mathcal{P}$-equivalence, since it is in fact a chain homotopy equivalence. Since C is $\mathcal{P}$-cofibrant, the map

$$\mathrm{Ch}(\mathcal{A})(C, D^{n+1}M) \to \mathrm{Ch}(\mathcal{A})(C, D^{n+1}N)$$

is surjective. But this map is isomorphic to the map $\mathcal{A}(C_n, M) \to \mathcal{A}(C_n, N)$, so C_n is relatively projective.

If K is weakly $\mathcal{P}$-contractible, then the natural map $PK \to K$ is a $\mathcal{P}$-trivial fibration. Since C is $\mathcal{P}$-cofibrant, any map $C \to K$ factors through PK, which means that it is chain homotopic to 0.

The converse follows immediately from Lemma 2.3, since the kernel of a $\mathcal{P}$-trivial fibration is weakly $\mathcal{P}$-contractible. □

Proposition 2.5. *A map $i\colon A \to B$ is a $\mathcal{P}$-cofibration if and only if i is a degreewise split monomorphism with $\mathcal{P}$-cofibrant cokernel.*

Proof. Suppose first that i is a $\mathcal{P}$-cofibration with cokernel C. Since $\mathcal{P}$-cofibrations are closed under pushouts, it is clear that C is $\mathcal{P}$-cofibrant. The map $D^{n+1}A_n \to 0$ is a $\mathcal{P}$-fibration and a $\mathcal{P}$-equivalence. Since i is a $\mathcal{P}$-cofibration, the map $A \to D^{n+1}A_n$ that is the identity in degree n extends to a map $B \to D^{n+1}A_n$. In degree n, this map defines a splitting of i_n.

Conversely, suppose that i is a degreewise split monomorphism and the cokernel C of i is $\mathcal{P}$-cofibrant. We need to show that i has the left lifting property with respect to any $\mathcal{P}$-trivial fibration $p\colon X \to Y$. But this follows by combining Lemmas 2.3 and 2.4. □

The next lemma provides a source of $\mathcal{P}$-cofibrant objects, including the $\mathcal{P}$-cellular complexes.

Definition 2.6. Call a complex C **purely $\boldsymbol{\mathcal{P}}$-cellular** if it is a colimit of a colimit-preserving diagram

$$0 = C^0 \to C^1 \to C^2 \to \cdots$$

indexed by an ordinal γ, such that for each $\alpha < \gamma$ the map $C^\alpha \to C^{\alpha+1}$ is degreewise split monic with cokernel a complex of relative projectives with zero differential. By "colimit-preserving" we mean that for each limit ordinal $\lambda < \gamma$, the map $\operatorname{colim}_{\alpha<\lambda} C^\alpha \to C^\lambda$ is an isomorphism. We say C is **$\boldsymbol{\mathcal{P}}$-cellular** if it is a retract of a purely $\mathcal{P}$-cellular complex.

Lemma 2.7.
(a) If D in $Ch(\mathcal{A})$ is a complex of relative projectives with zero differential, then D is $\mathcal{P}$-cofibrant.
(b) If D in $Ch(\mathcal{A})$ is a bounded below complex of relative projectives, then D is $\mathcal{P}$-cofibrant.
(c) If D in $Ch(\mathcal{A})$ is $\mathcal{P}$-cellular, then D is $\mathcal{P}$-cofibrant.

Proof. (a) follows immediately from Lemma 2.4.

(b) Let D be a bounded below complex of relative projectives and write $D^{\le n}$ for the truncation of D that agrees with D in degrees $\le n$ and is 0 elsewhere. Then the map $D^{\le n} \to D^{\le n+1}$ is degreewise split monic and has a $\mathcal{P}$-cofibrant cokernel (by (a)), so is a $\mathcal{P}$-cofibration by Proposition 2.5. Since $D^{\le n} = 0$ for $n << 0$ and 0 is $\mathcal{P}$-cofibrant, each $D^{\le n}$ is $\mathcal{P}$-cofibrant. Therefore, so is their colimit D.

(c) The proof is just a transfinite version of the proof of (b), combined with the fact that a retract of a cofibrant object is cofibrant. □

We can now prove that the other lifting axiom holds.

Proposition 2.8. *A map $i\colon A \to B$ has the left lifting property with respect to $\mathcal{P}$-fibrations if and only if i is a $\mathcal{P}$-trivial cofibration.*

Proof. Suppose first that i has the left lifting property with respect to $\mathcal{P}$-fibrations. Then i is a $\mathcal{P}$-cofibration, by definition, and the cokernel $0 \to C$ also has the left lifting property with respect to $\mathcal{P}$-fibrations. In particular, since the map $PC \to C$ is a $\mathcal{P}$-fibration, C is contractible. Hence i is a chain homotopy equivalence, and in particular a $\mathcal{P}$-trivial cofibration.

Conversely, suppose that i is a $\mathcal{P}$-trivial cofibration with cokernel C. By Lemma 2.3, in order to show that i has the left lifting property with respect to $\mathcal{P}$-fibrations, it suffices to show that every map from C to any complex K is chain homotopic to 0. This is equivalent to showing that C is contractible. Since i is degreewise split monic, for each relative projective P there is a long exact sequence $\cdots \to [\Sigma^k P, A] \to [\Sigma^k P, B] \to [\Sigma^k P, C] \to \cdots$. Since i is a $\mathcal{P}$-equivalence, $[\Sigma^k P, C] = 0$ for each relative projective P

and each k, and so $PC \to C$ is a $\mathcal{P}$-trivial fibration. Since C is $\mathcal{P}$-cofibrant, the identity map of C factors through PC, and so C is contractible. □

Note that the proof shows that a $\mathcal{P}$-trivial cofibration is in fact a chain homotopy equivalence.

Now we proceed to prove the factorization axioms, under the assumption that we have cofibrant replacements.

Proposition 2.9. *If every object A has a cofibrant replacement $q_A\colon QA \to A$, then every map in Ch($\mathcal{A}$) can be factored into a $\mathcal{P}$-cofibration followed by a $\mathcal{P}$-trivial fibration.*

Proof. Suppose $f\colon A \to B$ is a map in $\mathrm{Ch}(\mathcal{A})$. Let C be the cofibre of f, so $C = B \oplus \Sigma A$ with $d(b,a) = (db + fa, -da)$, and let E be the fibre of the composite $g\colon QC \to C \to \Sigma A$, so $E = A \oplus QC$ with $d(a,q) = (da - gq, dq)$ (the desuspension of the cofibre). Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, equal] & E \arrow[r] & QC \arrow[r, "g"] \arrow[d, "q_C"'] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r] & C \arrow[r] & \Sigma A \end{tikzcd}$$

whose rows are triangles in $\mathcal{K}(\mathcal{A})$. There is a natural fill-in map $p\colon E \to B$ defined by $p(a,q) = f(a) + \pi_B q_C q$, where $\pi_B\colon C \to B$ is the projection. The map p makes the left-hand square commute in $\mathrm{Ch}(\mathcal{A})$ and the middle square commute in $\mathcal{K}(\mathcal{A})$ (with the chain homotopy $s(a,q) = (0,a)$). The map $i\colon A \to E$ is a $\mathcal{P}$-cofibration since it is degreewise split and its cokernel QC is $\mathcal{P}$-cofibrant. Furthermore, since $QC \to C$ is degreewise $\mathcal{P}$-epic and $\pi_B\colon C \to B$ is degreewise split epic, it follows that p is degreewise $\mathcal{P}$-epic. Applying the functor $[\Sigma^k P, -]$ gives two long exact sequences, and from the five-lemma one sees that $[\Sigma^k P, p]$ is an isomorphism when P is a relative projective. Thus $f = pi$ is the required factorization. □

Proposition 2.10. *If every object A has a cofibrant replacement $q_A\colon QA \to A$, then every map in Ch($\mathcal{A}$) can be factored into a $\mathcal{P}$-trivial cofibration followed by a $\mathcal{P}$-fibration.*

Proof. It is well-known that we can factor any map $f\colon A \to B$ in $\mathrm{Ch}(\mathcal{A})$ into a degreewise split monomorphism that is also a chain homotopy equivalence, followed by a degreewise split epimorphism. Since every degreewise split epimorphism is a $\mathcal{P}$-fibration, we may as well assume f is a degreewise split monomorphism and a chain homotopy equivalence.

In this case, we apply Proposition 2.9 to factor $f = pi$, where p is a $\mathcal{P}$-trivial fibration and i is a $\mathcal{P}$-cofibration. Since f is a chain homotopy equivalence, i must be a $\mathcal{P}$-trivial cofibration, and so the proof is complete. □

Note that the factorizations constructed in Proposition 2.9 and Proposition 2.10 are both functorial in the map f, since we are implicitly assuming that cofibrant replacement is functorial.

The homotopy category of $\mathrm{Ch}(\mathcal{A})$, formed by inverting the $\mathcal{P}$-equivalences, is called the **derived category** of $\mathcal{A}$ (with respect to $\mathcal{P}$). It is denoted $\mathcal{D}(\mathcal{A})$, and a fundamental result in model category theory asserts that $\mathcal{D}(\mathcal{A})(X,Y)$ is a set for each X and Y. In Exercise 10.4.5 of [Wei94], Weibel outlines an argument that proves that $\mathcal{D}(\mathcal{A})(X,Y)$ is a set when there are enough (categorical) projectives, $\mathcal{P}$ is the categorical projective class, and $\mathcal{A}$ satisfies AB5. A connection between Weibel's hypotheses and Case B is that if $\mathcal{A}$ has enough projectives that are small with respect to all filtered diagrams in $\mathcal{A}$, then AB5 holds. The smallness condition needed for our theorem is weaker than this. (See Section 4 for the precise hypothesis.)

2.1. **Properties of the relative model structure.** In this subsection, we investigate some of the properties of the relative model structure. We begin by showing that the model category notions of homotopy, derived functor, suspension and cofibre sequence agree with the usual notions. Then we study properness and monoidal structure. We discuss cofibrant generation in Section 5.

We assume throughout that $\mathcal{P}$ is a projective class on an abelian category $\mathcal{A}$ such that the relative model structure on $\mathrm{Ch}(\mathcal{A})$ exists.

We first show that the notion of homotopy determined from the model category structure corresponds to the usual notion of chain homotopy.

Definition 2.11. ([DS95] or [Qui67].) If M is an object in a model category $\mathcal{C}$, a **good cylinder object** for M is an object $M \times I$ and a factorization $M \coprod M \xrightarrow{i} M \times I \xrightarrow{p} M$ of the codiagonal map, with i a cofibration and p a weak equivalence. (Despite the notation, $M \times I$ is *not* in general a product of M with an object I.) A **left homotopy** between maps $f, g \colon M \to N$ is a map $H \colon M \times I \to N$ such that the composite Hi is equal to $f \coprod g \colon M \coprod M \to N$, for some good cylinder object $M \times I$.

The notion of **good path object** N^I for N is dual to that of good cylinder object and leads to the notion of **right homotopy**. The following standard result can be found in [DS95, Section 4], for example.

Lemma 2.12. *For M cofibrant and N fibrant, two maps $f, g \colon M \to N$ are left homotopic if and only if they are right homotopic, and both of these relations are equivalence relations and respect composition. Moreover, if $M \times I$ is a fixed good cylinder object for M, then f and g are left homotopic if and only if they are left homotopic using $M \times I$; similarly for a fixed good path object.* □

Because of the lemma, for M cofibrant and N fibrant we have a well-defined relation of **homotopy** on maps $M \to N$. Quillen showed that the homotopy category of $\mathcal{C}$, which is by definition the category of fractions formed by inverting the weak equivalences, is equivalent to the category consisting of objects that are both fibrant and cofibrant with morphisms being homotopy classes of morphisms.

Now we return to the study of the model category $\mathrm{Ch}(\mathcal{A})$.

Lemma 2.13. *Let M and N be objects of $Ch(\mathcal{A})$ with M $\mathcal{P}$-cofibrant. Two maps $M \to N$ are homotopic if and only if they are chain homotopic.*

Proof. We construct a factorization $M \oplus M \to M \times I \to M$ of the codiagonal map $M \oplus M \to M$ in the following way. Let $M \times I$ be the chain complex that has $M_n \oplus M_{n-1} \oplus M_n$ in degree n. We describe the differential by saying that it sends a generalized element $(m, \bar{m}, m')$ in $(M \times I)_n$ to $(dm + \bar{m}, -d\bar{m}, dm' - \bar{m})$. Let $i \colon M \oplus M \to M \times I$ be the map that sends (m, m') to $(m, 0, m')$ and let $p \colon M \times I \to M$ be the map that sends $(m, \bar{m}, m')$ to $m + m'$. One can check easily that $M \times I$ is a chain complex and that i and p are chain maps whose composite is the codiagonal. The map i is degreewise split monic with cokernel ΣM, so it is a $\mathcal{P}$-cofibration, since we have assumed that M is cofibrant. The map p is a chain homotopy equivalence with chain homotopy inverse sending m to $(m, 0, 0)$; this implies that it induces a chain homotopy equivalence of generalized elements and is thus a $\mathcal{P}$-equivalence. Therefore $M \times I$ is a good cylinder object for M.

It is easy to see that a chain homotopy between two maps $M \to N$ is the same as a left homotopy using the good cylinder object $M \times I$. By Lemma 2.12, two maps are homotopic if and only if they are left homotopic using $M \times I$. Thus the model category notion of homotopy is the same as the notion of chain homotopy when the source is $\mathcal{P}$-cofibrant. □

There is a dual proof that proceeds by constructing a specific good path object N^I for N such that a right homotopy using N^I is the same as a chain homotopy.

Corollary 2.14. *Let A and B be objects of $\mathcal{A}$ considered as chain complexes concentrated in degree* 0*. Then $\mathcal{D}(\mathcal{A})(A, \Sigma^n B) \cong \operatorname{Ext}^n_{\mathcal{P}}(A, B)$.*

See Subsection 1.2 for the definition of the Ext groups.

Proof. The group $\mathcal{D}(\mathcal{A})(A, \Sigma^n B)$ may be calculated by choosing a $\mathcal{P}$-cofibrant replacement A' for A and computing the homotopy classes of maps from A' to $\Sigma^n B$. (Recall that all objects are $\mathcal{P}$-fibrant, so there is no need to take a fibrant replacement for $\Sigma^n B$.) A $\mathcal{P}$-resolution P of A serves as a $\mathcal{P}$-cofibrant replacement for A, and by Lemma 2.13 the homotopy relation on $\operatorname{Ch}(\mathcal{A})(P, \Sigma^n B)$ is chain homotopy, so it follows that $\mathcal{D}(\mathcal{A})(A, \Sigma^n B)$ is isomorphic to $\operatorname{Ext}^n_{\mathcal{P}}(A, B)$. □

More generally, a similar argument shows that the derived functors of a functor F can be expressed as the cohomology of the derived functor of F in the model category sense. To make the story complete, we next show that the shift functor Σ corresponds to the notion of suspension that the category $\mathcal{D}(\mathcal{A})$ obtains as the homotopy category of a pointed model category.

Definition 2.15. Let $\mathcal{C}$ be a pointed model category. If M is cofibrant, we define the **suspension** ΣM of M to be the cofibre of the map $M \coprod M \to M \times I$ for some good cylinder object $M \times I$. (The cofibre of a map $X \to Y$ is the pushout $* \coprod_X Y$, where $*$ is the zero object.) ΣM is cofibrant and well-defined up to homotopy equivalence.

The loop object ΩN of a fibrant object N is defined dually. These operations induce adjoint functors on the homotopy category. A straightforward argument based on the cylinder object described above (and a dual path object) proves the following lemma.

Lemma 2.16. *In the model category Ch($\mathcal{A}$), the functor Σ defined in Definition 2.15 can be taken to be the usual suspension, so that $(\Sigma X)_n = X_{n-1}$ and $d_{\Sigma X} = -d_X$. Similarly, ΩX can be taken to be the complex $\Sigma^{-1} X$. That is, $(\Omega X)_n = X_{n+1}$ and $d_{\Omega X} = -d_X$.* □

In particular, Σ and Ω are inverse functors. The second author [Hov98] has shown that this implies that cofibre sequences and fibre sequences agree (up to the usual sign) and that Σ and the cofibre sequences give rise to a triangulation of the homotopy category. (See [Qui67, Section I.3] for the definition of cofibre and fibre sequences in any pointed model category.) Using the explicit cylinder object from the proof of Lemma 2.13, we can be more explicit.

Corollary 2.17. *The category $\mathcal{D}(\mathcal{A})$ is triangulated with the usual suspension. A sequence $L \to M \to N \to \Sigma L$ is a triangle if and only if it is isomorphic in $\mathcal{D}(\mathcal{A})$ to the usual cofibre sequence on the map $L \to M$ (see Section 0).* □

Now we show that the model structures we construct are proper. A good reference for proper model categories is [Hir00, Chapter 11].

Proposition 2.18. *Let $\mathcal{P}$ be any projective class on an abelian category $\mathcal{A}$. Consider the commutative square in Ch($\mathcal{A}$) below.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

*(a) If the square is a pullback square, p is a $\mathcal{P}$-fibration, and g is a $\mathcal{P}$-equivalence, then f is a $\mathcal{P}$-equivalence. That is, the relative model structure is **right proper**.*

*(b) If the square is a pushout square, q is a degreewise split monomorphism, and f is a $\mathcal{P}$-equivalence, then g is a $\mathcal{P}$-equivalence. In particular, the pushout of a $\mathcal{P}$-equivalence along a $\mathcal{P}$-cofibration is a $\mathcal{P}$-equivalence. That is, the relative model structure is **left proper**.*

A model category that is both left and right proper is said to be **proper**. Note that we don't actually need to know that our $\mathcal{P}$-cofibrations, $\mathcal{P}$-fibrations and $\mathcal{P}$-equivalences give a model structure to ask whether the structure is proper.

Proof. Part (a) is an immediate consequence of [Hir00, Corollary 11.1.3], since every object is $\mathcal{P}$-fibrant. For part (b), let C be the cokernel of q. Since pushouts are computed degreewise, it follows that p is a degreewise split monomorphism with cokernel C. Thus we have a map of triangles

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \\ \downarrow & & \downarrow \\ C & \xrightarrow[\mathrm{id}]{} & C \end{array}$$

in the homotopy category $\mathcal{K}(\mathcal{A})$. The top and bottom maps are $\mathcal{P}$-equivalences, so the middle map must be as well, by using the five-lemma and the long exact sequences obtained by applying the functors $[\Sigma^k P, -]$. □

We now consider monoidal structure. Monoidal model categories are studied in [Hov98, Chapter 4]. We will assume that $\mathcal{A}$ is a closed monoidal category. Thus it is equipped with a functor $\otimes\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ such that both $A\otimes -$ and $-\otimes A$ have right adjoints for each A in $\mathcal{A}$. In particular, what we need is that these functors preserve colimits. There is, of course, an induced closed monoidal structure on $\mathrm{Ch}(\mathcal{A})$, for which we also use the notation $\otimes$.

Proposition 2.19. *Let A be a closed monoidal abelian category with a projective class $\mathcal{P}$ such that cofibrant replacements exist and the unit is $\mathcal{P}$-projective. Then the relative model structure on $\mathrm{Ch}(\mathcal{A})$ is monoidal if and only if the tensor product of two $\mathcal{P}$-cofibrant complexes is always $\mathcal{P}$-cofibrant.*

Proof. There are two conditions that must hold for a model category to be monoidal. One of them is automatically satisfied when the unit is cofibrant. The unit of the monoidal structure on $\mathrm{Ch}(\mathcal{A})$ is $\Sigma^0 S$, where S is the unit of $\mathcal{A}$, which has been assumed to be $\mathcal{P}$-projective. Thus the unit is $\mathcal{P}$-cofibrant, by Lemma 2.7 (a),

Therefore, the relative model structure is monoidal if and only if whenever $f\colon A\to B$ and $g\colon X\to Y$ are $\mathcal{P}$-cofibrations, then the map

$$f\,\square\, g\colon\ (A\otimes Y)\amalg_{A\otimes X}(B\otimes X)\to B\otimes Y$$

is a $\mathcal{P}$-cofibration, and is a $\mathcal{P}$-trivial cofibration if either f or g is a $\mathcal{P}$-trivial cofibration. It is easy to see that $f\,\square\, g$ is a degreewise split monomorphism, with cokernel $C\otimes Z$.

By taking f to be the map $0\to C$ and g to be the map $0\to Z$, we see that if the relative model structure is monoidal, then the tensor product of two $\mathcal{P}$-cofibrant complexes is $\mathcal{P}$-cofibrant. Conversely, if $C\otimes Z$ is $\mathcal{P}$-cofibrant whenever C and Z are $\mathcal{P}$-cofibrant, the preceding paragraph implies $f\,\square\, g$ is a $\mathcal{P}$-cofibration whenever f and g are $\mathcal{P}$-cofibrations. If either f or g is a $\mathcal{P}$-trivial cofibration, then one of C or Z is $\mathcal{P}$-trivially cofibrant, and hence contractible. It follows that $C\otimes Z$ is contractible, and so $f\,\square\, g$ is a $\mathcal{P}$-trivial cofibration. □

In particular, suppose the relative model structure is monoidal, and M, N are $\mathcal{P}$-projective. Then $\Sigma^0 M\otimes\Sigma^0 N\cong\Sigma^0(M\otimes N)$ is $\mathcal{P}$-cofibrant, and therefore $M\otimes N$ is $\mathcal{P}$-projective. With this in mind, we say that a projective class $\mathcal{P}$ on a closed monoidal category $\mathcal{A}$ is **monoidal** if the unit is $\mathcal{P}$-projective and the tensor product of $\mathcal{P}$-projectives is $\mathcal{P}$-projective.

We do not know of any example of a monoidal projective class $\mathcal{P}$ where cofibrant replacements exist, but the relative model structure is not monoidal. Certainly this does not happen in either Case A or Case B of Theorem 2.2. In Case A, we have the following corollary.

Corollary 2.20. *Suppose $F\colon \mathcal{B} \to \mathcal{A}$ is a monoidal functor between closed monoidal abelian categories, with right adjoint U that preserves countable coproducts. Then the relative model structure on $Ch(\mathcal{A})$ is monoidal.*

Proof. Proposition 3.3 says that $QX \otimes QY$ is $\mathcal{P}$-cofibrant for any X and Y, where Q denotes the cofibrant replacement functor constructed in Section 3. If X and Y are already cofibrant, then lifting implies that X is a retract of QX and Y is a retract of QY. Thus $X \otimes Y$ is a retract of $QX \otimes QY$, so $X \otimes Y$ is $\mathcal{P}$-cofibrant. Proposition 2.19 completes the proof. $\square$

In Case B, we will show in Corollary 4.4 that every $\mathcal{P}$-cofibrant object is $\mathcal{P}$-cellular. Thus the following corollary applies.

Corollary 2.21. *Let $\mathcal{A}$ be a closed monoidal abelian category with a monoidal projective class $\mathcal{P}$ such that cofibrant replacements exist and every $\mathcal{P}$-cofibrant object is $\mathcal{P}$-cellular (Definition 2.6). Then the relative model structure on $Ch(\mathcal{A})$ is monoidal.*

Proof. Let A and B be $\mathcal{P}$-cofibrant. By Proposition 2.19, it suffices to show that $A \otimes B$ is $\mathcal{P}$-cofibrant. By assumption, A is a retract of a transfinite colimit of a colimit-preserving diagram

$$0 = A^0 \to A^1 \to \cdots$$

such that for each α, the map $A^\alpha \to A^{\alpha+1}$ is degreewise split monic with cokernel a complex of relative projectives with zero differential. Since $- \otimes B$ preserves retracts, degreewise split monomorphisms and colimits, it is enough to prove that $\Sigma^i P \otimes B$ is $\mathcal{P}$-cofibrant for each $\mathcal{P}$-projective P and each i. Applying the same filtration argument to B, we find that it suffices to show that $\Sigma^i P \otimes \Sigma^j Q = \Sigma^{i+j}(P \otimes Q)$ is $\mathcal{P}$-cofibrant for all $\mathcal{P}$-projectives P and Q and integers i and j. But this follows immediately from the fact that $\mathcal{P}$ is monoidal and Lemma 2.7 (a). $\square$

We can also prove a dual statement in Case A.

Proposition 2.22. *Suppose $U\colon \mathcal{A} \to \mathcal{B}$ is a monoidal functor of closed monoidal abelian categories, with right adjoint F. Assume that U preserves countable products. Then the injective relative model structure on $Ch(\mathcal{A})$ is monoidal.*

By the "injective relative model structure", we mean the model structure obtained by dualizing Theorem 2.2. The (trivial) cofibrations in this model structure are called the **$\mathcal{B}$-injective (trivial) cofibrations**.

Proof. Suppose $f\colon A \to B$ and $g\colon X \to Y$ are $\mathcal{B}$-injective cofibrations with cokernels C and Z, respectively. This means that Uf and Ug are degreewise split monomorphisms. Recall the definition of $f \,\square\, g$ used in the proof of Proposition 2.19. Since U is monoidal and preserves pushouts, $U(f \,\square\, g) \cong Uf \,\square\, Ug$. One can easily check that $Uf \,\square\, Ug$ is a degreewise split monomorphism, so $f \,\square\, g$ is a $\mathcal{B}$-injective cofibration. If f is a $\mathcal{B}$-injective trivial cofibration, then the cokernel C of f has UC contractible. Let Z denote the cokernel of g. Since the cokernel of $Uf \,\square\, Ug$ is $UC \otimes UZ$, which is contractible, $f \,\square\, g$ is a $\mathcal{B}$-injective trivial cofibration. Since everything is cofibrant in the injective relative model structure, the other condition in the definition of a monoidal model category is automatically satisfied. $\square$

Another important property of a model category is whether it is cofibrantly generated. This is the topic of Section 5.

2.2. **Why projective classes?** The astute reader will notice that we haven't used the assumption that there is always a $\mathcal{P}$-epimorphism from a $\mathcal{P}$-projective. Indeed, all we have used is that we have a collection $\mathcal{P}$ of objects such that, with the definitions at the beginning of this section, cofibrant replacements exist. The following proposition explains why we start with a projective class.

Proposition 2.23. *Let $\mathcal{P}$ be a collection of objects in an abelian category $\mathcal{A}$ such that cofibrant replacements exist in Ch($\mathcal{A}$). Then there is a unique projective class $(\mathcal{P}', \mathcal{E})$ that gives rise to the same definitions of weak equivalence, fibration and cofibration.*

Proof. Let $\mathcal{E}$ be the collection of $\mathcal{P}$-epimorphisms and let $\mathcal{P}'$ be the collection of all objects P such that every map in $\mathcal{E}$ is P-epic. Then $\mathcal{P}'$ contains $\mathcal{P}$, and a map is $\mathcal{P}'$-epic if and only if it is $\mathcal{P}$-epic. It follows that the $\mathcal{P}'$-exact sequences are the same as the $\mathcal{P}$-exact sequences. A map f is a $\mathcal{P}$-equivalence if and only if the cofibre of f is $\mathcal{P}$-exact. The same holds for the $\mathcal{P}'$-equivalences, and since the two notions of exactness also agree, the two notions of equivalence agree. Finally, since cofibrations are defined in terms of the fibrations and weak equivalences, the two notions of cofibration agree.

That the pair $(\mathcal{P}', \mathcal{E})$ is a projective class follows immediately from the existence of cofibrant replacements.

Our work earlier in this section shows that the objects of $\mathcal{P}'$ are precisely those objects P that are cofibrant when viewed as complexes concentrated in degree 0. Thus the projective class is the unique projective class giving rise to the same weak equivalences, fibrations and cofibrations. □

Thus by requiring our collection of objects to be part of a projective class, we in effect choose a canonical collection of objects determining each model structure we produce. This means that the relevant question is: which projective classes give rise to model structures? Theorem 2.2 gives a necessary and sufficient condition, namely the existence of cofibrant replacements, and we know of no projective classes that do not satisfy this condition.

An additional advantage of having a projective class is that it provides us with the language to state results such as: $\mathrm{Ext}^n_{\mathcal{P}}(M, N) \cong \mathrm{HoCh}(\mathcal{A})(\Sigma^n M, N)$ (see Corollary 2.14).

3. Case A: Projective classes coming from adjoint pairs

In this section we prove Case A of Theorem 2.2.

Let $U\colon \mathcal{A} \to \mathcal{B}$ be a functor of abelian categories, with a left adjoint $F\colon \mathcal{B} \to \mathcal{A}$. Then U and F are additive, U is left exact and F is right exact. Let $\mathcal{P}$ be the projective class on $\mathcal{A}$ that is the pullback of the trivial projective class on $\mathcal{B}$ (see Example 1.3 and Subsection 1.3). In this subsection we construct a cofibrant replacement functor for the projective class $\mathcal{P}$ under the assumption that U preserves countable coproducts.

Because this projective class is strong, Lemma 1.9 tells us that the $\mathcal{P}$-fibrations and $\mathcal{P}$-equivalences have alternate characterizations. A map f in Ch($\mathcal{A}$) is a $\mathcal{P}$-fibration if and only if $\mathcal{A}(P, f)$ is degreewise *split* epic for each P in $\mathcal{P}$, and is a $\mathcal{P}$-equivalence if and only if $\mathcal{A}(P, f)$ is a *chain homotopy equivalence* for each P in $\mathcal{P}$.

First we prove a lemma characterizing the fibrations and weak equivalences in this model structure and giving us a way to generate cofibrant objects.

Lemma 3.1. *(a) A map $p\colon X \to Y$ is a $\mathcal{P}$-fibration if and only if Up is a degreewise split epimorphism in Ch($\mathcal{B}$).*
(b) A map $p\colon X \to Y$ is a $\mathcal{P}$-equivalence if and only if Up is a chain homotopy equivalence in Ch($\mathcal{B}$).
(c) If i is a map in Ch($\mathcal{B}$) that is degreewise split monic, then Fi is a $\mathcal{P}$-cofibration.
(d) For any C in Ch($\mathcal{B}$), FC is $\mathcal{P}$-cofibrant.

Proof. For part (a), note that the $\mathcal{P}$-projectives are all retracts of FM for $M \in \mathcal{B}$. Hence p is a $\mathcal{P}$-fibration if and only if $\mathcal{A}(FM, p) = \mathcal{B}(M, Up)$ is a surjection for all $M \in \mathcal{B}$. This is true if and only if Up is a degreewise split epimorphism. For part (b), a similar argument shows that p is a $\mathcal{P}$-equivalence if and only if $\mathcal{B}(M, Up)$ is a quasi-isomorphism for all $M \in \mathcal{B}$. We claim that this forces Up to be a chain homotopy equivalence. Indeed, let C denote the cofiber of Up. Then $\mathcal{B}(M, C)$ is exact for all $M \in \mathcal{B}$. By taking $M = Z_nC$, we find that C is exact and that $C_{n+1} \to Z_nC$ is a split epimorphism. It follows that C is contractible, as in the proof of Lemma 1.9. Thus Up is a chain homotopy equivalence.

For part (c), let C be the cokernel of i. Then Fi is degreewise split monic with cokernel FC. Suppose that $p\colon X \to Y$ is a $\mathcal{P}$-trivial fibration with kernel K. Then K is $\mathcal{P}$-trivially fibrant, so part (b) implies that UK is contractible. By Lemma 2.3, to show that Fi is a $\mathcal{P}$-cofibration, it suffices to show that every chain map $FC \to \Sigma K$ is chain homotopic to 0. By adjointness, it suffices to show that every chain map $C \to \Sigma UK$ is chain homotopic to 0. Since UK is contractible, this is clear. Part (d) follows from part (c), □

We now provide a construction, given a complex X, of a $\mathcal{P}$-cofibrant complex QX and a $\mathcal{P}$-trivial fibration $QX \to X$. This comes out of the bar construction, which we now recall from [ML95, Section IX.6]. Given an object N of $\mathcal{A}$, define the complex BN by $(BN)_m = (FU)^{m+1}N$ when $m \geq -1$ and 0 otherwise. We will define maps $s = s_m\colon U(BN)_{m-1} \to U(BN)_m$ and $\delta = \delta_m\colon (BN)_m \to (BN)_{m-1}$. For $m < 0$ we declare both to be zero. For $m \geq 0$, $s_m\colon U(BN)_{m-1} = (UF)^m UN \to U(BN)_m = UF(UF)^m UN$ is defined to be adjoint to the identity map of $F(UF)^m UN$. For $m \geq 0$, we can then inductively define $\delta_m\colon (BN)_m = FU(FU)^m N \to (BN)_{m-1} = (FU)^m N$ to be adjoint to the self-map $1 - s_{m-1}(U\delta_{m-1})$ of $U(FU)^m N$. Properties of adjoint functors then guarantee that $(U\delta_m)s_m + s_{m-1}(U\delta_{m-1}) = 1$. Using this and the properties of adjoint functors, one deduces a sequence of implications $\delta_{m-1}\delta_m = 0 \implies (U\delta_m)s_m(U\delta_m) = U\delta_m \implies (U\delta_m)(U\delta_{m+1})s_{m+1} = 0 \implies \delta_m\delta_{m+1} = 0$. Therefore $\delta_{m-1}\delta_m = 0$ for each m and so δ makes BN into a chain complex.

The construction of BN, δ, and s is obviously natural in N. Thus, given a complex X in $\mathrm{Ch}(\mathcal{A})$, we get a bicomplex $(BX)_{m,n} = (BX_n)_m$, where

$$\delta\colon (BX)_{m,n} \to (BX)_{m-1,n} \text{ and } d\colon (BX)_{m,n} \to (BX)_{m,n-1}$$

commute. Furthermore, $s\colon U(BX)_{m,n} \to U(BX)_{m+1,n}$ and Ud also commute.

We can then define a total complex $\overline{Q}X$ by $(\overline{Q}X)_k = \bigoplus_{m+n=k}(BX)_{m,n}$. The differential ∂ of $\overline{Q}X$ takes the summand $(BX)_{m,n}$ into $(BX)_{m-1,n} \oplus (BX)_{m,n-1}$ by $(\delta, (-1)^m d)$. Note that $\overline{Q}X$ is a filtered complex, with $F^i\overline{Q}X$ the subcomplex consisting of terms with $m \leq i$. In particular, $F^{-1}\overline{Q}X = \Sigma^{-1}X$. We define QX to be $\overline{Q}X/\Sigma^{-1}X$, so that $(QX)_k = (FU)X_k \oplus (FU)^2 X_{k-1} \oplus \cdots$.

The following proposition proves Case A of Theorem 2.2.

Proposition 3.2. *There is a natural $\mathcal{P}$-trivial fibration $q_X\colon QX \to X$, and QX is $\mathcal{P}$-cofibrant.*

Proof. We first show that QX is $\mathcal{P}$-cofibrant. There is an increasing filtration $\{F^iQX\}_{i\geq 0}$ on QX, where $F^0QX = FUX$ and $F^iQX/F^{i-1}QX = (FU)^{i+1}\Sigma^i X$. Furthermore, each inclusion $F^iQX \to F^{i+1}QX$ is a degreewise split monomorphism. By Lemma 3.1 (d), each quotient $(FU)^{i+1}\Sigma^i X$ is $\mathcal{P}$-cofibrant, so each map $F^{i-1}QX \to F^iQX$ is a $\mathcal{P}$-cofibration. Hence $FUX \to QX$ is a $\mathcal{P}$-cofibration. Thus QX is $\mathcal{P}$-cofibrant.

The map q_X is induced by $\delta_0\colon (FU)X_n \to X_n$, adjoint to the identity. The map q_X sends the other summands of $(QX)_n$ to 0. We leave to the reader the check that this is a chain map. Since $U\delta_0$ is a split epimorphism, Uq_X is a degreewise split epimorphism, so is a $\mathcal{P}$-fibration by Lemma 3.1 (a). To show q_X is a $\mathcal{P}$-equivalence, it suffices to show that the fiber $\overline{Q}X$ is $\mathcal{P}$-contractible, or, equivalently, that $U\overline{Q}X$ is contractible (Lemma 3.1 (b)). The contracting homotopy is given by s. Indeed, on the summand $U(BX)_{m,n}$, the $U(BX)_{m,n}$ component of $s(U\partial) + (U\partial)s$ is $s(U\delta) + (U\delta)s = 1$, and the $U(BX)_{m+1,n-1}$ component is $s(-1)^m(Ud) + (-1)^{m+1}(Ud)s = 0$. (This is where we use that U commutes with coproducts.) □

The following proposition implies that the relative model structure is monoidal in this case, as explained in Corollary 2.20.

Proposition 3.3. *Suppose $F\colon \mathcal{B} \to \mathcal{A}$ is a monoidal functor of closed monoidal abelian categories, with right adjoint U that preserves countable coproducts. Then, for any X and Y in $Ch(\mathcal{A})$, $QX \otimes QY$ is $\mathcal{P}$-cofibrant.*

Proof. Recall the filtration F^iQX on QX used in the proof of Proposition 3.2. Using this filtration, we find that $QX \otimes QY$ is the colimit of $F^iQX \otimes QY$, and each map $F^{i-1}QX \otimes QY \to F^iQX \otimes QY$ is a degreewise split monomorphism with cokernel $(FU)^{i+1}\Sigma^i X \otimes QY$. It therefore suffcies to show that this cokernel is $\mathcal{P}$-cofibrant for all i. A similar argument using the filtration on QY shows that it suffices to show that

$$(FU)^{i+1}(\Sigma^i X) \otimes (FU)^{j+1}(\Sigma^j Y) \cong F(U(FU)^i\Sigma^i X \otimes U(FU)^j\Sigma^j Y)$$

is $\mathcal{P}$-cofibrant for all $i, j \geq 0$. But this follow immediately from Lemma 3.1 (d). □

3.1. **Examples.**

Example 3.4. Let $\mathcal{B}$ be a bicomplete abelian category. Since the identity functor is adjoint to itself and preserves coproducts, we can apply Theorem 2.2 to the trivial projective class $\mathcal{P}$ to conclude that $\mathrm{Ch}(\mathcal{B})$ is a model category. We call this the **absolute model structure**. The $\mathcal{P}$-equivalences are the chain homotopy equivalences (Lemma 3.1), the $\mathcal{P}$-fibrations are the degreewise split epimorphisms and the $\mathcal{P}$-cofibrations are the degreewise split monomorphisms. Every object is both $\mathcal{P}$-cofibrant and $\mathcal{P}$-fibrant, and the homotopy category is the usual homotopy category $\mathcal{K}(\mathcal{B})$ in which chain homotopic maps have been identified. That this model structure exists was also shown in [Col]. Note that if $\mathcal{B}$ is closed monoidal, the absolute model structure is also monoidal, by Corollary 2.20. In particular, since every object is $\mathcal{P}$-cofibrant, given a differential graded algebra $R \in \mathrm{Ch}(\mathcal{B})$, we get a model structure on the category of differential graded R-modules, where weak equivalences are chain homotopy equivalences (of the underlying chain complexes) and fibrations are degreewise split epimorphisms.[1]

Now let $\mathcal{A}$ and $\mathcal{B}$ be bicomplete abelian categories and let $U\colon \mathcal{A} \to \mathcal{B}$ be a coproduct preserving functor with left adjoint F. By Theorem 2.2, $\mathrm{Ch}(\mathcal{A})$ has a relative model structure whose weak equivalences, cofibrations and fibrations are called $\mathcal{B}$-equivalences, $\mathcal{B}$-cofibrations and $\mathcal{B}$-fibrations. This structure is a lifting of the absolute model structure on $\mathrm{Ch}(\mathcal{B})$ in the sense that a map f in $\mathrm{Ch}(\mathcal{A})$ is a weak equivalence or fibration if and only if Uf is so in $\mathrm{Ch}(\mathcal{B})$. It is often the case that one wants to lift a model structure along a right adjoint. When the model structure is cofibrantly generated, necessary and sufficient conditions for a lifting are known [DHK97, 9.1], [Hir00, 13.4.2]. Our main theorem says that it is also possible to lift the absolute model structure on $\mathrm{Ch}(\mathcal{B})$, even though it is not usually cofibrantly generated (see Subsection 5.4).

It follows from the above that F preserves cofibrations and trivial cofibrations. The adjoint functors F and U form a Quillen pair.

The category $\mathrm{Ch}(\mathcal{A})$ also has an absolute model structure. The identity functor sends absolute fibrations and weak equivalences to $\mathcal{B}$-fibrations and $\mathcal{B}$-equivalences. Thus the identity functor is a right Quillen functor from the absolute model structure to the relative model structure.

[1] This is not obvious; see notes on last page.

Example 3.5. Let $R \to S$ be a map of rings. Write R-mod and S-mod for the categories of left R- and S-modules. Consider the forgetful functor $U\colon S\text{-mod} \to R\text{-mod}$ and its left adjoint F that sends an R-module M to $S \otimes_R M$. We saw in Example 1.6 that this gives a projective class whose relative projectives are the S-modules P such that the natural map $S \otimes_R M \to M$ is split epic as a map of S-modules. The functor U preserves coproducts, so Theorem 2.2 and Lemma 1.9 tell us that $\mathrm{Ch}(S) = \mathrm{Ch}(S\text{-mod})$ has a model structure in which the weak equivalences are the maps that become chain homotopy equivalences after forgetting the S-module structure. The fibrations are the maps that in each degree are split epic as maps of R-modules. The cofibrations are defined by the left lifting property with respect to the trivial fibrations. Equivalently, by Proposition 2.5, they are the degreewise split monomorphisms whose cokernels are cofibrant. And by Lemma 2.4, a complex C is cofibrant if and only if each C_n is a relative projective and every map from C to a complex K such that UK is contractible is chain homotopic to 0. Lemma 2.7 gives us a ready supply of cofibrant objects. In particular, a relative resolution of an S-module M is a cofibrant replacement, so the group $\mathrm{HoCh}(S)(M, \Sigma^i N)$ of maps in the homotopy category is isomorphic to the relative Ext^i group [ML95].

When R and S are commutative, the functor F is monoidal, so this relative model structure is monoidal by Corollary 2.20. There is then a derived tensor product $X \otimes^L Y$ in $\mathrm{HoCh}(S)$, and if M and N are S-modules, $H_i(M \otimes^L N)$ is isomorphic to the relative Tor_i group [ML95].

Example 3.6. Again let $R \to S$ be a map of rings and let U be the forgetful functor. We saw in Example 1.7 that U has a right adjoint G that sends an R-module M to $R\text{-mod}(S, M)$ and so we get an injective class on S-mod by pulling back the trivial injective class along U. U preserves products, so we can apply the duals of Theorem 2.2 and Lemma 1.9 to conclude that $\mathrm{Ch}(S)$ has a model structure with the same weak equivalences as in the previous example. The cofibrations are the maps that in each degree are split monic as maps of R-modules, and the fibrations are the maps with the right lifting property with respect to with respect to the trivial cofibrations. Fibrations and fibrant objects can also be characterized by the duals of Lemmas 2.4 and 2.7, and the homotopy category again encodes the relative Ext groups. However, this relative model structure is not monoidal, even when R and S are commutative, since U is not monoidal.

Example 3.7. Suppose A is an algebra over a commutative ring k, and let $\mathcal{C}$ denote the category of chain complexes of A-bimodules. Then there are forgetful functors from A-bimodules to left A-modules, right A-modules, and k-modules. Each of these preserves coproducts, so we get three different relative model structures on $\mathcal{C}$. When we forget to right or left A-modules, the cofibrant replacement functor Q applied to A gives us the usual un-normalized bar construction, with $(QA)_n = A^{\otimes n+2}$. The complex QA is then also cofibrant in the relative model structure obtained by forgetting to k-modules, since it is a bounded below complex of relative projectives. The Hochschild cohomology $HH^n(A; N)$ of A with coefficients in a bimodule N (or a chain complex of bimodules N) is then equal to $\mathrm{Ho}\mathcal{C}(\Sigma^n A, N)$, where we can use any of the three relative model structures above. We can extend this definition by replacing A with an arbitrary complex of bimodules, but then the answer may depend on which relative model structure we use. It is most natural to use the relative model structure obtained from the forgetful functor to k-modules, as then we can define Hochschild homology as well.

Indeed, the Hochschild homology groups are defined using the tensor product $M \otimes N$ of bimodules, where we identify $ma \otimes n$ with $m \otimes an$ but also $am \otimes n$ with $m \otimes na$. We emphasize that $M \otimes N$ is only a k-module, not a bimodule. In particular, this tensor product can't be associative, but it is commutative and also unital in a weak sense, since $M \otimes (A \otimes A) \cong M$ as a k-module. Furthermore, the functor $- \otimes N$ has a right adjoint that takes a k-module L to the bimodule $\mathrm{Hom}_k(N, L)$, where, if $g \in \mathrm{Hom}(N, L)$, $(ga)(n) = g(an)$ and $(ag)(n) = g(na)$. Both of these functors then extend to functors on chain complexes. The proof of Proposition 2.19 applies to this case as well, and shows that if f and g are cofibrations in the relative model structure on $\mathcal{C}$ obtained by forgetting to k-modules, then $f \square g$ is a cofibration in the absolute model structure on chain complexes of k-modules. If either f or g is a trivial cofibration, so is $f \square g$. This means the tensor product has a total left derived functor that is commutative and unital (in the above weak sense). We can therefore extend the usual definition of Hochschild homology to complexes X and Y of bimodules, by defining $HH_n(X, Y) = H_n(QX \otimes QY)$. In fact, it is possible to prove, using the technique of Proposition 2.19, that, if X is cofibrant, the functor $X \otimes -$ takes weak equivalences in the relative model structure to chain homotopy equivalences. This implies that $HH_n(X, Y) = H_n(QX \otimes Y)$. This is a direct generalization of the way Hochschild homology is defined in [ML95, Section X.4].

4. Case B: Projective classes with enough small projectives

In this section we prove that cofibrant replacements exist in Case B. We begin by introducing the terminology necessary for the precise statement of Case B.

We think of an ordinal as the set of all previous ordinals, and of a cardinal as the first ordinal with that cardinality.

Definition 4.1. Given a limit ordinal γ, the **cofinality** of γ, $\operatorname{cofin}\gamma$, is the smallest cardinal κ such that there exists subset T of γ of cardinality κ with $\sup T = \gamma$. The cofinality of a successor ordinal is defined to be 1.

A **colimit-preserving sequence** from an ordinal γ to a category $\mathcal{A}$ is a diagram

$$X^0 \to X^1 \to \cdots \to X^\alpha \to X^{\alpha+1} \to \cdots$$

of objects of $\mathcal{A}$ indexed by the ordinals less than γ, such that for each limit ordinal λ less than γ the natural map $\operatorname{colim}_{\alpha<\lambda} X^\alpha \to X^\lambda$ is an isomorphism.

For κ a cardinal, an object P is said to be **κ-small relative to a subcategory $\mathcal{M}$** if for each ordinal γ with $\operatorname{cofin}\gamma > \kappa$ and each colimit-preserving sequence $X \colon \gamma \to \mathcal{A}$ that factors through $\mathcal{M}$, the natural map $\operatorname{colim}_{\alpha<\gamma} \mathcal{A}(P, X^\alpha) \to \mathcal{A}(P, \operatorname{colim}_{\alpha<\gamma} X^\alpha)$ is an isomorphism, where the last colimit is taken in $\mathcal{A}$.

If $\mathcal{P}$ is a projective class and $\mathcal{P}' \subseteq \mathcal{P}$, we say that the relative projectives in $\mathcal{P}'$ are **enough** if every $\mathcal{P}'$-epimorphism is a $\mathcal{P}$-epimorphism. That is, the collection $\mathcal{P}'$ is enough to check $\mathcal{P}$-epimorphisms, $\mathcal{P}$-fibrations and $\mathcal{P}$-equivalences. We also sometimes say that $\mathcal{P}$ is determined by $\mathcal{P}'$.

Fix a cardinal κ and let $\mathcal{P}'$ be the collection of all $\mathcal{P}$-projectives that are κ-small relative to the subcategory of $\mathcal{A}$ consisting of split monomorphisms with $\mathcal{P}$-projective cokernel. We say that $\mathcal{P}$ has **enough κ-small projectives** if the collection $\mathcal{P}'$ is enough.

The following proposition proves Case B of Theorem 2.2.

Proposition 4.2. *Let $\mathcal{P}$ be a projective class on a complete and cocomplete abelian category $\mathcal{A}$. Assume that $\mathcal{P}$-resolutions can be chosen functorially, and that for some cardinal κ there are enough κ-small projectives. Then functorial cofibrant replacements exist.*

Note that we do not assume that there is a *set* of relative projectives that is enough. The proof below is a variant of the small object argument that avoids needing a set of test objects by using the properties of projective classes.

Proof. Let X be a chain complex. The basic construction is the following "partial cofibrant replacement": For each i choose $\mathcal{P}$-epimorphisms $P_i \to X_i$ and $Q_i \to Z_i X$ with P_i and Q_i $\mathcal{P}$-projective. Then form $\bigoplus_i (D^i P_i \oplus \Sigma^i Q_i)$, which has a natural map to

X. This map is degreewise $\mathcal{P}$-epic (because of the P_i's) and is epic under $\mathrm{Ch}(\mathcal{A})(\Sigma^k P, -)$ for any $\mathcal{P}$-projective P (because of the Q_i's). And its domain is $\mathcal{P}$-cofibrant.

We now construct a transfinite sequence $D^1 \to D^2 \to \cdots$ with maps to X. Define $D_1 \to X$ to be a partial cofibrant replacement. Let C^1 be the pullback

$$\begin{array}{ccc} C^1 & \longrightarrow & D^1 \\ \big\downarrow & & \big\downarrow \\ PX & \longrightarrow & X, \end{array}$$

where PX is the path complex of X. Let $P^1 \to C^1$ be a partial cofibrant replacement. Define D^2 to be the cofibre of the composite $P^1 \to C^1 \to D^1$. The map $D^1 \to D^2$ is a degreewise split monomorphism whose cokernel (ΣP^1) is $\mathcal{P}$-cofibrant, so it is a $\mathcal{P}$-cofibration. Moreover, since the map $P^1 \to C^1 \to D^1 \to X$ factors through PX, there is a canonical null homotopy, and so there is a canonical map $D^2 \to X$.

Let γ be an ordinal with cofinality greater than κ, and inductively define $D^\alpha \to X$ for $\alpha < \gamma$. When α is a limit ordinal, D^α is defined to be the colimit of the earlier D^β. Let D be the colimit of all of the D^α. D is $\mathcal{P}$-cofibrant and comes with a map to X.

Since $D^1 \to X$ is already degreewise $\mathcal{P}$-epic, so is $D \to X$. That is, $D \to X$ is a $\mathcal{P}$-fibration.

Let P be a relative projective that is κ-small relative to the split monomorphisms with relative projective cokernels. We need to show that $D \to X$ is sent to an isomorphism by $[\Sigma^k P, -]$. To simplify notation, we do the case where $k = 0$. That $[P, D] \to [P, X]$ is epic is clear since $\mathrm{Ch}(\mathcal{A})(P, D^1) \to \mathrm{Ch}(\mathcal{A})(P, X)$ is already so. So all that remains is to show that $[P, D] \to [P, X]$ is monic. Let $P \to D$ be a chain map such that $P \to D \to X$ is null. By Lemma 4.3 below, the chain map $P \to D$ factors as a chain map through some D^β. It is null in X, so the composite $P \to X$ factors through PX. Thus we get a chain map to the pullback C^β. And this lifts through $P^\beta \to C^\beta$. So the composite $P \to D^\beta \to D^{\beta+1}$ must be null. In particular, the original map $P \to D$ must be null. $\square$

We owe the reader a proof of the following lemma.

Lemma 4.3. *Let P be an object of $\mathcal{A}$ that is κ-small relative to a subcategory $\mathcal{M}$ of $\mathcal{A}$. Then the chain complex $\Sigma^k P$ is κ-small relative to the subcategory of Ch($\mathcal{A}$) consisting of maps whose components are in $\mathcal{M}$.*

Proof. Suppose D is the colimit of a γ-indexed diagram whose maps $D^\alpha \to D^{\alpha+1}$ have components in $\mathcal{M}$, where γ has cofinality greater than κ. For simplicity we treat the case $k = 0$. Suppose we have a chain map $P \to D$. The map $f\colon P \to D_0$ factors through some D_0^α, since P is small and the maps $D_0^\alpha \to D_0^{\alpha+1}$ are in $\mathcal{M}$. Then $df\colon P \to D_{-1}^\alpha$ goes to zero in D, and so goes to zero in some D^β, using the other half of smallness. So the chain map $P \to D$ factors as a chain map through some D^β. This shows that $\operatorname{colim} \mathrm{Ch}(\mathcal{A})(P, D^\alpha) \to \mathrm{Ch}(\mathcal{A})(P, D)$ is surjective. That it is injective is equivalent to the fact that $\operatorname{colim} \mathrm{Ch}(\mathcal{A})(P, D_0^\alpha) \to \mathrm{Ch}(\mathcal{A})(P, D_0)$ is injective. $\square$

Corollary 4.4. *In the situation of Proposition 4.2, every $\mathcal{P}$-cofibrant complex is $\mathcal{P}$-cellular.*

Proof. It suffices to show that the cofibrant replacement D constructed in Proposition 4.2 is $\mathcal{P}$-cellular, since if X is cofibrant, a lifting argument shows that X is a retract of any cofibrant replacement of X. The complex D is constructed as a colimit of a colimit-preserving functor D^α, indexed by an ordinal γ. Each map $D^\alpha \to D^{\alpha+1}$ is a degreewise split monomorphism with cokernel that is easily seen to be purely $\mathcal{P}$-cellular. By reindexing, one can show that the colimit of such a transfinite diagram is (purely) $\mathcal{P}$-cellular. $\square$

5. Cofibrant generation

In this section we assume that $\mathcal{A}$ is a complete and cocomplete abelian category whose objects are small (Definition 4.1) and we characterize the projective classes on $\mathcal{A}$ that give rise to cofibrantly generated model structures. We also show that for the model structures we put on $\mathrm{Ch}(\mathcal{A})$, cofibrant generation is equivalent to having a set of weak generators. Our goal is the following theorem.

Theorem 5.1. *Let $\mathcal{A}$ be a complete and cocomplete abelian category whose objects are small and let $\mathcal{P}$ be any projective class on $\mathcal{A}$. Then the following are equivalent:*

(*i*) *$\mathcal{P}$ is determined by a set* (*Subsection 5.2*).
(*ii*) *The $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations, and $\mathcal{P}$-cofibrations form a model structure on Ch($\mathcal{A}$), and this model structure is cofibrantly generated* (*Subsection 5.1*).
(*iii*) *The $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations, and $\mathcal{P}$-cofibrations form a model structure on Ch($\mathcal{A}$), and the associated homotopy category has a set of weak generators* (*Subsection 5.4*).

Proof. That (i) $\implies$ (ii) is proved in Subsection 5.2.
That (ii) $\implies$ (iii) is [Hov98, Theorem 7.3.1].
That (iii) $\implies$ (i) is proved in Subsection 5.4. □

In Subsection 5.3 we give some examples that are cofibrantly generated, and in Subsection 5.4 we give some examples that are not cofibrantly generated.

5.1. **Background.** In this section we briefly recall the basics of cofibrantly generated model categories. This material will be used in the next section to prove Theorem 5.1. For more details, see the books by Dwyer, Hirschhorn and Kan [DHK97], Hirschhorn [Hir00], and the second author [Hov98, Section 2.1]. We will always assume our model categories to be complete and cocomplete. See Definition 4.1 for the definition of "small".

Definition 5.2. Let I be a class of maps in a cocomplete category. A map is said to be ***I*-injective** if it has the right lifting property with respect to each map in I, and we write I-inj for the category containing these maps. A map is said to be an ***I*-cofibration** if it has the left lifting property with respect to each map in I-inj, and we write I-cof for the category containing these maps. A map is said to be ***I*-cellular** if it is a transfinite composite of pushouts of coproducts of maps in I, and we write I-cell for the category containing these maps.

Note that I-cell is a subcategory of I-cof.

Definition 5.3. A **cofibrantly generated model category** is a model category $\mathcal{M}$ for which there exist sets I and J of morphisms with domains that are small relative to I-cof and J-cof, respectively, such that I-cof is the category of cofibrations and J-cof is the category of trivial cofibrations. It follows that I-inj is the category of trivial fibrations and that J-inj is the category of fibrations.

For example, take $\mathcal{M}$ to be the category of spaces and take $I = \{S^n \to B^{n+1}\}$ and $J = \{B^n \times 0 \to B^n \times [0,1]\}$.

Proposition 5.4. (Recognition Lemma.) *Let $\mathcal{M}$ be a category that is complete and cocomplete, let W be a class of maps that is closed under retracts and satisfies the two-out-of-three axiom, and let I and J be sets of maps with domains that are small relative to I-*cell *and J-*cell*, respectively, such that*

(*i*) J-cell $\subseteq I$-cof $\cap\, W$ *and* I-inj $\subseteq J$-inj $\cap\, W$, *and*
(*ii*) J-cof $\supseteq I$-cell $\cap\, W$ *or* I-inj $\supseteq J$-inj $\cap\, W$.

Then $\mathcal{M}$ is cofibrantly generated by I and J, and W is the subcategory of weak equivalences. Moreover, a map is in I-cof if and only if it is a retract of a map in I-cell, and similarly for J-cof and J-cell. □

The proof, which uses the small object argument and is due to Kan, can be found in [DHK97], [Hir00], and [Hov98, Theorem 2.1.19].

5.2. **Projective classes with sets of enough small projectives.** In this subsection we prove that under a hypothesis slightly stronger than Case B of Theorem 2.2 we can conclude that the model structure is cofibrantly generated. We will prove this from scratch, since the additional work is small, thanks to the theory of cofibrantly generated model categories.

Let $\mathcal{A}$ be a complete and cocomplete abelian category equipped with a projective class $\mathcal{P}$. We use the terminology from Section 0 and the refer the reader to Definition 2.1 for the definitions of $\mathcal{P}$-equivalence, $\mathcal{P}$-fibration and $\mathcal{P}$-cofibration.

The following lemma will be used to prove our main result.

Lemma 5.5. *Consider an object $P \in \mathcal{A}$ and a map $f\colon X \to Y$ of chain complexes.*

- (*i*) *The map f has the right lifting property with respect to each of the maps $0 \to D^n P$, $n \in \mathbb{Z}$, if and only if the induced map of P-elements is a surjection.*
- (*ii*) *The map f has the right lifting property with respect to each of the maps $\Sigma^{n-1}P \to D^n P$, $n \in \mathbb{Z}$, if and only if the induced map of P-elements is a surjection and a quasi-isomorphism.*

We use the following terminology in the proof. The cycles, boundaries and homology classes in $\mathcal{A}(P, X)$ are called ***P*-cycles**, ***P*-boundaries** and ***P*-homology classes** in X. Note that a P-cycle is the same as a P-element of $Z_n X$, but that not every P-element of $B_n X$ is necessarily a P-boundary. A P-element is just a chain map $D^n P \to X$, a P-cycle is just a chain map $\Sigma^n P \to X$, with P regarded as a complex concentrated in degree 0, and a P-homology class is just a chain homotopy class of maps $\Sigma^n P \to X$.

Proof. We begin with (i): The map f has the right lifting property with respect to the map $0 \to D^n P$ if and only if each map $D^n P \to Y$ factors through f, *i.e.,* if and only if each P-element of Y_n is in the image of f_n.

Now (ii): The map f has the right lifting property with respect to the map $\Sigma^{n-1}P \to D^n P$ if and only if for each P-element y of Y_n whose boundary is the image of a P-cycle x of X_{n-1}, there is a P-element x' of X_n that hits y under f and x under the differential. (In other words, if and only if the natural map $X_n \to Z_{n-1}X \times_{Z_{n-1}Y} Y_n$ induces a surjection of P-elements.)

So suppose that f has the right lifting property with respect to each map $\Sigma^{n-1}P \to D^n P$. As a preliminary result, we prove that f induces a surjection of P-cycles. Suppose we are given a P-cycle y of Y. Its boundary is zero and is thus the image of the P-cycle 0 of X. Therefore y is the image of a P-cycle x'.

It follows immediately that f induces a surjection in P-homology.

Now we prove that f induces a surjection of P-elements. Suppose we are given a P-element y of Y. By the above argument, its boundary, which is a P-cycle, is the image of a P-cycle x of X. Thus, by the characterization of maps f having the RLP, we see that there is a P-element x' that hits y.

A similar argument shows that f induces an injection in P-homology.

We have proved that if f has the right lifting property with respect to the maps $\Sigma^{n-1}P \to D^n P$, then f induces an isomorphism in P-homology and a surjection of P-elements. The proof of the converse goes along the same lines. □

Corollary 5.6. *A map $f\colon X \to Y$ is a $\mathcal{P}$-fibration (resp. $\mathcal{P}$-trivial fibration) if and only if it has the right lifting property with respect to the map $0 \to D^n P$ (resp. $\Sigma^{n-1}P \to D^n P$) for each $\mathcal{P}$-projective P and each $n \in \mathbb{Z}$.* □

We want to claim that the above definitions lead to a cofibrantly generated model category structure on the category $\mathrm{Ch}(\mathcal{A})$. In order to prove this, we need to assume that there is a *set* $\mathcal{S}$ of $\mathcal{P}$-projectives such that a map $f\colon A \to B$ is $\mathcal{P}$-epic if and only if f induces a surjection of P-elements for each P in $\mathcal{S}$. This implies that for any B there is a $\mathcal{P}$-epimorphism $P \to B$ with P a coproduct of objects from $\mathcal{S}$, and that every $\mathcal{P}$-projective object is a retract of such a coproduct (cf. Lemma 1.5). We also need to assume that each P in $\mathcal{S}$ is small relative to the subcategory K of split monomorphisms with $\mathcal{P}$-projective cokernels. (See Section 4 for the definition of "small.") When these conditions hold we say that $\mathcal{P}$ is **determined by a set of small objects** or that there is a **set of enough small projectives**.

Theorem 5.7. *Assume that $\mathcal{P}$ is determined by a set $\mathcal{S}$ of small objects. Then the category $Ch(\mathcal{A})$ is a cofibrantly generated model category with the following generating sets:*

$$I := \{\Sigma^{n-1}P \to D^nP \mid P \in \mathcal{S},\, n \in \mathbb{Z}\},$$
$$J := \{0 \to D^nP \mid P \in \mathcal{S},\, n \in \mathbb{Z}\}.$$

The weak equivalences, fibrations and cofibrations are the $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations and $\mathcal{P}$-cofibrations as described in Definition 2.1. A map is a $\mathcal{P}$-cofibration if and only if it is degreewise split monic with $\mathcal{P}$-cellular cokernel.

In particular, every object is $\mathcal{P}$-fibrant and an object is $\mathcal{P}$-cofibrant if and only if it is $\mathcal{P}$-cellular (Definition 2.6).

Proof. We check the hypotheses of the Recognition Lemma from the previous subsection. Since $\mathcal{A}$ is complete and cocomplete, so is $\mathrm{Ch}(\mathcal{A})$; limits and colimits are taken degreewise. The class W of $\mathcal{P}$-equivalences is easily seen to be closed under retracts and to satisfy the two-out-of-three condition. The zero chain complex is certainly small relative to J-cell. It is easy to see that a map is in I-cell if and only if it is a degreewise split monomorphism whose cokernel is purely cellular (Definition 2.6). In particular, every map in I-cell has components in $K \subseteq \mathcal{A}$. We assumed that each P in $\mathcal{S}$ is small relative to K, and so it follows from Lemma 4.3 that each $\Sigma^k P$ is small relative to I-cell $\subseteq \mathrm{Ch}(\mathcal{A})$.

Since the projectives in $\mathcal{S}$ are enough to test whether a map is a $\mathcal{P}$-fibration or a $\mathcal{P}$-equivalence, Corollary 5.6 tells us that I-inj is the collection of $\mathcal{P}$-trivial fibrations and that J-inj is the collection of $\mathcal{P}$-fibrations. Thus we have an equality $I\text{-inj} = J\text{-inj} \cap W$, giving us two of the inclusions required by the Recognition Lemma.

We now prove that $J\text{-cell} \subseteq I\text{-cof} \cap W$. Since $I\text{-inj} \subseteq J\text{-inj}$, it is clear that $J\text{-cof} \subseteq I\text{-cof}$, so in particular $J\text{-cell} \subseteq I\text{-cof}$. We must prove that $J\text{-cell} \subseteq W$, *i.e.,* that each map that is a transfinite composite of pushouts of coproducts of maps in J is a $\mathcal{P}$-equivalence. A map in J is of the form $0 \to D^nP$ for some $P \in \mathcal{S}$. Thus a pushout of a coproduct of maps in J is of the form $X \to X \oplus C$, with C a contractible complex, and a transfinite composite of such maps is of the same form as well. Thus such a map is in fact a chain homotopy equivalence, so it is certainly a $\mathcal{P}$-equivalence.

We can now apply the Recognition Lemma and conclude that $\mathrm{Ch}(\mathcal{A})$ is a model category with weak equivalences the $\mathcal{P}$-equivalences. The fibrations are the maps in I-inj, which, as we noted above, are the $\mathcal{P}$-fibrations. The cofibrations are the maps in I-cof, *i.e.,* the maps with the left lifting property with respect to the $\mathcal{P}$-trivial fibrations, and this is precisely how the $\mathcal{P}$-cofibrations were defined.

The recognition lemma tells us that the $\mathcal{P}$-cofibrations consist precisely of the retracts of maps in I-cell. As discussed above, I-cell is the class of degreewise split monomorphisms with purely cellular cokernels. So the $\mathcal{P}$-cofibrations are the degreewise split monomorphisms whose cokernels are cellular. $\square$

5.3. **The pure and categorical derived categories.** In this section we let R be an associative ring with unit and we take for $\mathcal{A}$ the category of left R-modules. We are concerned with two projective classes on the category $\mathcal{A}$. The first is the **categorical projective class** $\mathcal{C}$ whose projectives are summands of free modules, whose exact sequences are the usual exact sequences, and whose epimorphisms are the surjections. The second is the **pure projective class** $\mathcal{P}$ whose projectives are summands of sums of finitely presented modules. A short exact sequence is $\mathcal{P}$-exact iff it remains exact after tensoring with any right module. A map is a $\mathcal{P}$-epimorphism iff it appears as the epimorphism in a $\mathcal{P}$-exact short exact sequence. We say **pure projective** instead of $\mathcal{P}$-projective, and similarly for **pure exact** and **pure epimorphism**. We assume that the reader has some familiarity with these projective classes. A brief summary with further references may be found in [Chr98, Section 9]. As usual, we write $\mathrm{Ext}^*(-, B)$ (resp. $\mathrm{PExt}^*(-, B)$) for the derived functors of $\mathcal{A}(-, B)$ with respect to the categorical (resp. pure) projective class.

Both of these projective classes are determined by sets of small objects: $\mathcal{C}$ is determined by $\{R\}$ and $\mathcal{P}$ is determined by any set of finitely presented modules containing a representative from each isomorphism class. Thus we get two cofibrantly generated model category structures on $\mathrm{Ch}(R)$ and two derived categories, the categorical derived category $\mathcal{D}_{\mathcal{C}}$ and the pure derived category $\mathcal{D}_{\mathcal{P}}$, both containing a set of weak generators. We refer to the pure weak equivalences in $\mathrm{Ch}(R)$ as **pure quasi-isomorphisms**, and as usual call the categorical weak equivalences simply quasi-isomorphisms. Similarly, we talk of pure fibrations and fibrations, pure cofibrations and cofibrations, etc.

Pure homological algebra is of interest in stable homotopy theory because of the following result.

Theorem 5.8. [CS98] *Phantom maps from a spectrum X to an Eilenberg-Mac Lane spectrum HG are given by $\mathrm{PExt}^1_{\mathbb{Z}}(H_{-1}X, G)$, that is, by maps of degree one from $H_{-1}X$ to G in the pure derived category of abelian groups.*

In addition to the connection between phantom maps and pure homological algebra, the authors are interested in the pure derived category as a tool for connecting the global pure dimension of a ring R to the behaviour of phantom maps in $\mathcal{D}_{\mathcal{C}}$ and $\mathcal{D}_{\mathcal{P}}$ under composition.

The two derived categories are connected in various ways. For example, since every categorical projective is pure projective, it follows that every pure quasi-isomorphism is a quasi-isomorphism. This implies that there is a unique functor $R : \mathcal{D}_{\mathcal{P}} \to \mathcal{D}_{\mathcal{C}}$ commuting with the functors from $\mathrm{Ch}(R)$. This functor has a left adjoint L. We can see this in the following way: the identity functor on $\mathrm{Ch}(R)$ is adjoint to itself. It is easy to see that every pure fibration is a fibration and that every pure trivial fibration is a trivial fibration. Therefore, by [DS95, Theorem 9.7], there is an induced pair of adjoint functors between the pure derived category and the categorical derived category, and the right adjoint is the functor R mentioned above. This right adjoint is the identity on objects, since everything is fibrant, and induces the natural map $\mathrm{PExt}^*(A, B) \to \mathrm{Ext}^*(A, B)$ for R-modules A and B. The left adjoint sends a complex X in $\mathcal{D}_{\mathcal{C}}$ to a (categorical) cofibrant replacement $\tilde{X}$ for X. Similar adjoint functors exist whenever one has two projective classes on a category, one containing the other.

5.4. **Failure to be cofibrantly generated.** In this subsection, we show that many of the model structures we have constructed are not cofibrantly generated and that their homotopy categories do not have a set of weak generators (see below). In particular, we show this for the absolute model structures on $\mathrm{Ch}(\mathbb{Z})$ and $\mathrm{Ch}(\mathbb{Z}_{(p)})$. Recall that the weak equivalences in these model structures are the chain homotopy equivalences, the cofibrations are the degreewise split monomorphisms, and the fibrations are the degreewise split epimorphisms.

A set $\mathcal{G}$ of objects in an additive category $\mathcal{H}$ is a **set of weak generators** if each non-zero object X in $\mathcal{H}$ receives a non-zero map from some G in $\mathcal{G}$.

A number of people have recently found proofs that certain model categories are not cofibrantly generated. The first such proof we have heard of is due to Dan Isaksen (personal communication), who proved that his model structure on pro-simplicial sets [Isa99] is not cofibrantly generated. In addition, Adámek, Herrlich, Rosický, and Tholen [AHRT00] have constructed another non-cofibrantly generated model structure. Furthermore, Neeman has proved [Nee01, Lemma E.3.2] that the triangulated category $\mathcal{K}(\mathbb{Z})$ does not have a set of weak generators. This implies that the absolute model structure on $\mathrm{Ch}(\mathbb{Z})$ is not cofibrantly generated, by [Hov98, Theorem 7.3.1].

We also prove that our model structures are not cofibrantly generated by showing that their homotopy categories do not have sets of weak generators. And we prove the latter using the following result, which completes the proof of Theorem 5.1.

Theorem 5.9. *Let $\mathcal{A}$ be a complete and cocomplete abelian category whose objects are small and let $\mathcal{P}$ be any projective class on $\mathcal{A}$. If the $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations, and $\mathcal{P}$-cofibrations form a model structure on Ch($\mathcal{A}$), and the associated homotopy category has a set of weak generators, then $\mathcal{P}$ is determined by a set of small objects.*

For the term "small," see Definition 4.1. In our examples, $\mathcal{A}$ will be the category R-mod for some ring R, and in this category every object is small.

Proof. Let $\mathcal{G}$ be a set of weak generators for $\mathrm{HoCh}(\mathcal{A})$ and, without loss of generality, assume that each G in $\mathcal{G}$ is $\mathcal{P}$-cofibrant. Let $\mathcal{S} = \{G_n \mid n \in \mathbb{Z} \text{ and } G \in \mathcal{G}\}$. Then each S in $\mathcal{S}$ is $\mathcal{P}$-projective. Suppose $p\colon X \to Y$ in $\mathcal{A}$ is $\mathcal{S}$-epic. We must show that p is $\mathcal{P}$-epic. Consider the chain complex $\ker p \to X \to Y$, with zeroes elsewhere. Because the map p is $\mathcal{S}$-epic, it is easy to check that every map from a generator G to the complex $\ker p \to X \to Y$ is null homotopic. But since $\mathcal{G}$ is a set of weak generators, this implies that this complex is $\mathcal{P}$-equivalent to the zero complex. That is, the complex $0 \to \mathcal{A}(P, \ker p) \to \mathcal{A}(P,X) \to \mathcal{A}(P,Y) \to 0$ is exact for each P in $\mathcal{P}$. In particular, the map $X \to Y$ is $\mathcal{P}$-epic. □

We now give examples of projective classes that are not determined by a set. We begin with a lemma about abelian groups.

Lemma 5.10. *For any cardinal κ, there is an abelian group A such that A is not a retract of any direct sum of abelian groups of cardinality less than κ.*

Proof. We use the Ulm invariants, as described in [Kap69]. Recall that for each prime p, each abelian group A and each ordinal λ, the Ulm invariant $U_p(A,\lambda)$ is a cardinal number. From the definition it is clear that $U_p(A,\lambda) = 0$ for λ larger than the cardinality of A. Furthermore, $U_p(-,\lambda)$ takes direct sums of abelian groups to sums of cardinals [Kap69, Problem 31]. Hence we only need to find an abelian group A with $U_p(A,\kappa) \neq 0$. Such a (p-torsion) group exists for every p by [Kap69, Problem 43]. □

Recall now the trivial projective class $\mathcal{P}$ on a category $\mathcal{A}$, which has all objects $\mathcal{P}$-projective and only the split epimorphisms $\mathcal{P}$-epic.

Lemma 5.11. *The trivial projective class on the category of abelian groups is not determined by a set.*

Proof. For any set $\mathcal{S}$ of abelian groups, we must exhibit a map $p\colon X \to Y$ that is $\mathcal{S}$-epic but not split epic. So fix a set $\mathcal{S}$ and let κ be a cardinal larger than the cardinality of each group in $\mathcal{S}$. Let Y be an abelian group that is not a retract of any direct sum of abelian groups of cardinality less than κ, using Lemma 5.10. Let X denote the direct sum of the C in $\mathcal{S}$, with one copy of C for each homomorphism $C \to Y$. Then there is an obvious map $p\colon X \to Y$, and this map is not split epic since Y is not a retract of

X. However, if B is an abelian group in $\mathcal{S}$, any homomorphism $f\colon B \to Y$ obviously factors through p. □

From this lemma and Theorem 5.1 we deduce:

Corollary 5.12. *The absolute model structure on $Ch(\mathbb{Z})$ is not cofibrantly generated, and the homotopy category $\mathcal{K}(\mathbb{Z})$ does not have a set of weak generators.* □

The results above generalize. Call a ring R **difficult** if the trivial projective class on the category of R-modules is not determined by a set. We showed above that $\mathbb{Z}$ is difficult, and it seems to be the case that many rings are difficult. For example, every proper subring R of $\mathbb{Q}$ is difficult. Indeed, the proof of Lemma 5.10 goes through with p chosen to be a prime that is not invertible in R. Then the proof of Lemma 5.11 applies to show that R is difficult. Similarly, one can show that polynomial rings are difficult, by using an indeterminate in place of the prime p. Moreover, we have the following lemma.

Lemma 5.13. *If R is a difficult ring and $R \to S$ is a map of rings that is split monic as a map of R-modules, then S is difficult. In addition, the* relative *projective class on S-mod is not determined by a set.*

Proof. Fix a cardinal κ. Since R is difficult, there is a map p of R-modules that is not split epic such that $\mathrm{Hom}(B, p)$ is epic whenever B has cardinality less than κ. Let q be the map $\mathrm{Hom}_R(S, p)$ of S-modules. Since R splits off of S, q is the sum of p and another map as a map of R-modules. Thus q is not split epic as an R-module map and hence as an S-module map. Since $\mathrm{Hom}_R(S, -)$ is right adjoint to the forgetful functor, we have $\mathrm{Hom}_S(M, q) = \mathrm{Hom}_R(M, p)$ for any S-module M. In particular, $\mathrm{Hom}_S(M, q)$ is surjective for all S-modules M of cardinality less than κ. Since any set of S-modules will have sizes bounded above by some cardinal, we have shown that the ring S is difficult.

Next we show that the relative projective class $\mathcal{P}$ on S-mod is also not determined by a set. Recall that the $\mathcal{P}$-projectives are the retracts of extended modules $M \otimes_R S$ and the $\mathcal{P}$-epimorphisms are the maps of S-modules that are split epic as R-module maps. So for each κ we must exhibit a map q of S-modules that is not split epic as an R-module map such that, for every extended module $M \otimes_R S$ of cardinality less than κ, the map $\mathrm{Hom}_S(M \otimes_R S, q)$ is epic. Note that neither requirement uses the S-module structure on q. We choose q as in the previous paragraph, where we already noted that it is not split epic as an R-module map. Since $\mathrm{Hom}_S(N, q)$ is surjective for all S-modules N of cardinality less than κ, of course $\mathrm{Hom}_S(M \otimes_R S, q)$ is surjective for all extended modules $M \otimes_R S$ of cardinality less than κ. □

On the other hand, if R is a semisimple ring, or equivalently, if every R-module is projective, then the trivial projective class is the same as the categorical projective class, and so is determined by a set (the set $\{R\}$). It follows that the absolute model structure on $\mathrm{Ch}(R)$ is cofibrantly generated and that $\mathcal{K}(R) = \mathcal{D}(R)$ has a set of weak generators. The hypotheses hold for fields and the rings $\mathbb{Z}/n$, where n is a product of distinct primes, for example. For other n, the ring $\mathbb{Z}/n$ is not semisimple, but it is easy to see that the trivial projective class is again determined by a set.

6. Simplicial objects and the bounded below derived category

In this section, we discuss the notion of a projective class on a pointed category and prove that under certain conditions the category of simplicial objects has a model structure reflecting a given projective class. We use this to deduce that for any projective class on any complete and cocomplete abelian category, the category of bounded below chain complexes has a model structure reflecting the projective class. We do not need to assume that our projective class comes from an adjoint pair or that there are enough small projectives. We also deduce that the category of G-equivariant simplicial sets has a model structure for any group G and any family of subgroups.

6.1. **Projective classes in pointed categories.** One can define the notion of a projective class on any pointed category by replacing the use of epimorphisms with exact sequences. However, in our applications we will assume that our pointed categories are bicomplete; in particular, they will have kernels, and when this is in the case, the definition given in Section 1 is equivalent to the more general definition [EM65]. So in what follows, a projective class on a pointed category is a collection $\mathcal{P}$ of objects and a collection $\mathcal{E}$ of maps satisfying Definition 1.1.

We supplement the examples from Section 1 with the following non-additive examples.

Example 6.1. Let $\mathcal{A}$ be the category of pointed sets, let $\mathcal{P}$ be the collection of all pointed sets and let $\mathcal{E}$ be the collection of all epimorphisms. Then $\mathcal{E}$ consists of the $\mathcal{P}$-epimorphisms and these classes form a projective class.

Example 6.2. More generally, if $\mathcal{A}$ is any pointed category with kernels, $\mathcal{P}$ is the collection of all objects, and $\mathcal{E}$ is the collection of all split epimorphisms, then $\mathcal{P}$ is a projective class. It is called the **trivial projective class**.

Example 6.1 is determined by a set in the sense of Lemma 1.5.

As in the abelian case, a projective class is precisely the information needed to form projective resolutions and define derived functors. However, in the non-additive case not all of the usual results hold.

For further details we refer the reader to the classic reference [EM65].

6.2. **The model structure.** Let $\mathcal{A}$ be a complete and cocomplete pointed category with a projective class $\mathcal{P}$. We do not assume that $\mathcal{A}$ is abelian. Write $s\mathcal{A}$ for the category of simplicial objects in $\mathcal{A}$. Given a simplicial object X and an object P, write $\mathcal{A}(P,X)$ for the simplicial set that has $\mathcal{A}(P,X_n)$ in degree n. Define a map f to be a **$\mathcal{P}$-equivalence** (resp. **$\mathcal{P}$-fibration**) if $\mathcal{A}(P,f)$ is a weak equivalence (resp. fibration) of simplicial sets for each P in $\mathcal{P}$. Define f to be a **$\mathcal{P}$-cofibration** if it has the left lifting property with respect to the $\mathcal{P}$-trivial fibrations.

Consider the following conditions:

- (*) For each X in $s\mathcal{A}$ and each P in $\mathcal{P}$, $\mathcal{A}(P,X)$ is a fibrant simplicial set.
- (**) $\mathcal{P}$ is determined by a set $\mathcal{S}$ of small objects. Here we require that each P in $\mathcal{S}$ be small[2] with respect to all split monomorphisms in $\mathcal{A}$, not just the split monomorphisms with $\mathcal{P}$-projective cokernels.

Theorem 6.3. *Let $\mathcal{A}$ be a complete and cocomplete pointed category with a projective class $\mathcal{P}$. If $\mathcal{A}$ and $\mathcal{P}$ satisfy (*) or (**), then the $\mathcal{P}$-equivalences, $\mathcal{P}$-fibrations and $\mathcal{P}$-cofibrations form a simplicial model structure. When (**) holds, this model structure is cofibrantly generated by the sets*

$$I := \{P \otimes \dot{\Delta}[n] \to P \otimes \Delta[n] \mid P \in \mathcal{S},\, n \geq 0\},$$
$$J := \{P \otimes V[n,k] \to P \otimes \Delta[n] \mid P \in \mathcal{S},\, 0 < n \geq k \geq 0\}.$$

In the above, $\Delta[n]$ denotes the standard n-simplex simplicial set; $\dot{\Delta}[n]$ denotes its boundary; and $V[n,k]$ denotes the subcomplex with the n-cell and its kth face removed. For an object P of $\mathcal{A}$ and a set K, $P \otimes K$ denotes the coproduct of copies of P indexed by K. When K is a simplicial set, $P \otimes K$ denotes the simplicial object in $\mathcal{A}$ that has $P \otimes K_n$ in degree n.

If $\mathcal{A}$ is abelian, then (*) holds. Moreover, in this case we have the Dold-Kan equivalence given by the normalization functor $s\mathcal{A} \to \mathrm{Ch}^+(\mathcal{A})$, where $\mathrm{Ch}^+(\mathcal{A})$ denotes the category of non-negatively graded chain complexes of objects of $\mathcal{A}$. Thus we can deduce:

Corollary 6.4. *Let $\mathcal{A}$ be a complete and cocomplete abelian category with a projective class $\mathcal{P}$. Then $\mathrm{Ch}^+(\mathcal{A})$ is a model category. A map f is a weak equivalence iff $\mathcal{A}(P,f)$ is a quasi-isomorphism for each P in $\mathcal{P}$. A map f is a fibration iff $\mathcal{A}(P,f)$ is surjective in*

[2]We in fact need that the objects in $\mathcal{S}$ are ω-small; see last page.

positive degrees (but not necessarily in degree 0) for each P in $\mathcal{P}$. A map is a cofibration iff it is degreewise split monic with degreewise $\mathcal{P}$-projective cokernel. Every complex is fibrant, and a complex is cofibrant iff it is a complex of $\mathcal{P}$-projectives. □

Note that no conditions on the projective class are required, and that the description of the cofibrations is simpler than in the unbounded case.

Special cases of the theorem and corollary were proved by Blanc [Bla99], with the hypothesis that the $\mathcal{P}$-projectives are cogroup objects. (Under this hypothesis, condition (*) automatically holds.)

As another example of the theorem, let G be a group and consider pointed G-simplicial sets, or equivalently, simplicial objects in the category $\mathcal{A}$ of pointed (say, left) G-sets. Let $\mathcal{F}$ be a family of subgroups of G, and consider the set $\mathcal{S} = \{(G/H)_+ \mid H \in \mathcal{F}\}$ of homogeneous spaces with disjoint basepoints. These are small, and determine a projective class. Thus, using case (**) of the above theorem, we can deduce:

Corollary 6.5. *Let G be a group*[3] *and let $\mathcal{F}$ be a family of subgroups of G. The category of pointed G-equivariant simplicial sets has a model category structure in which the weak equivalences are precisely the maps that induce a weak equivalence on H-fixed points for each H in $\mathcal{F}$.* □

We omit the proof of Theorem 6.3, and simply note that it follows the argument in [Qui67, Section II.4] fairly closely. It is a bit simpler, in that Quillen spends part of the time (specifically, his Proposition 2) proving that effective epimorphisms give rise to a projective class, although he doesn't use this terminology. It is also a bit more complicated, in the (**) case, in that a transfinite version of Kan's Ex^∞ functor [Kan57] is required, since we make a weaker smallness assumption. Quillen's argument in this case can be interpreted as a verification of the hypotheses of the recognition lemma for cofibrantly generated model categories (our Proposition 5.4).

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[3] G should be finite; see last page.

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Notes added after publication

We indicate here some corrections to the published version. We have not updated the body of the paper except to add footnotes which point the reader here.

At the end of the first paragraph of Example 3.4, we claim without proof that there is a model structure on the category of dg modules over a dg algebra, where weak equivalences are chain homotopy equivalences (of the underlying chain complexes) and fibrations are degreewise split epimorphisms. This is not at all clear, but will be explained in an upcoming paper by Tobias Barthel, Peter May and Emily Riehl. Thanks to Peter May and Tobias Barthel for pointing out this error.

In condition (**) used in Theorem 6.3, we should have required that the objects of $\mathcal{S}$ be ω-small with respect to all split monomorphisms in $\mathcal{A}$, not just small with respect to some cardinal. (In this form, the (**) case of Theorem 6.3 can also be found as Theorem II.5.6 in Paul G. Goerss, John F. Jardine, Simplicial Homotopy Theory, Birkhäuser, 1999.) This also means that the group G in Corollary 6.5 needs to be finite. Thanks to Theo Bühler for pointing out this error.

Hirschhorn's book [Hir00] has now been published as: Hirschhorn, Philip S. *Model categories and their localizations.* Mathematical Surveys and Monographs, 99. American Mathematical Society, Providence, RI, 2003.

The book [DHK97] has now been published as: Dwyer, William G.; Hirschhorn, Philip S.; Kan, Daniel M.; Smith, Jeffrey H. *Homotopy limit functors on model categories and homotopical categories.* Mathematical Surveys and Monographs, 113. American Mathematical Society, Providence, RI, 2004.

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