

Higher Toda brackets and the Adams spectral sequence

Dan Christensen
University of Western Ontario

Joint work with Martin Frankland

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Outline:

- Triangulated categories and injective classes
- The Adams spectral sequence
- 3-fold Toda brackets, and the relation to d_2
- Higher Toda brackets, and the relation to d_r
- 3-fold Toda brackets determine the rest

Triangulated categories

A **triangulated category** is an additive category \mathcal{T} equipped with an equivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$, and with a specified collection of **triangles** of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X. \quad (1)$$

These must satisfy the following axioms motivated by (co)fibre sequences in topology.

TR0: The triangles are closed under isomorphism.

The following is a triangle:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X.$$

TR1: Every map $X \rightarrow Y$ is part of a triangle (1).

TR2: (1) is a triangle iff (2) is a triangle:

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y. \quad (2)$$

Triangulated categories, II

\mathcal{T} additive, $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ an equivalence.

TR0: Triangles are closed under isomorphism and contain the trivial triangle.

TR1: Every map appears in a triangle.

TR2: Triangles can be rotated.

TR3: Given a solid diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow u & & \downarrow & & \downarrow \text{dotted} & & \downarrow \Sigma u \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

in which the rows are triangles, the dotted fill-in exists making the two squares commute.

TR4: The octahedral axiom holds. (Some details later.)

Examples and consequences

Example. The homotopy category of spectra.

Example. The derived category of a ring.

Example. The stable module category of a group algebra.

Example. The homotopy category of any stable Quillen model category.

Consequences: (1) For any object A , the sequences

$$\cdots \longrightarrow \mathcal{T}(A, X) \longrightarrow \mathcal{T}(A, Y) \longrightarrow \mathcal{T}(A, Z) \longrightarrow \mathcal{T}(A, \Sigma X) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow \mathcal{T}(X, A) \longleftarrow \mathcal{T}(Y, A) \longleftarrow \mathcal{T}(Z, A) \longleftarrow \mathcal{T}(\Sigma X, A) \longleftarrow \cdots$$

are exact sequences of abelian groups.

(2) The triangle containing a map $X \rightarrow Y$ is unique up to (non-unique) isomorphism.

Projective and injective classes

Eilenberg and Moore (1965) gave a framework for homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following:

Definition. A **projective class** in \mathcal{T} is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \text{ob } \mathcal{T}$ and $\mathcal{N} \subseteq \text{mor } \mathcal{T}$, such that:

- (i) \mathcal{P} consists of exactly the objects P such that each composite $P \rightarrow X \rightarrow Y$ is zero for each $X \rightarrow Y$ in \mathcal{N} ,
- (ii) \mathcal{N} consists of exactly the maps $X \rightarrow Y$ such that each composite $P \rightarrow X \rightarrow Y$ is zero for each P in \mathcal{P} ,
- (iii) for each X in \mathcal{T} , there is a triangle $P \rightarrow X \rightarrow Y$ with P in \mathcal{P} and $X \rightarrow Y$ in \mathcal{N} .

The first two conditions are easy to satisfy. The third says that there are enough projectives.

An **injective class** in \mathcal{T} is a projective class in \mathcal{T}^{op} .

Examples of projective and injective classes

Example. In spectra, take \mathcal{P} to be all retracts of wedges of spheres and \mathcal{N} to consist of all maps inducing the zero map in homotopy groups. Then $(\mathcal{P}, \mathcal{N})$ is a projective class.

The analogous construction works starting with any set of objects in any triangulated category with all coproducts.

Example. Dually, if E is any spectrum, take \mathcal{I} to be all retracts of products of suspensions of E and \mathcal{N} to consist of all maps inducing the zero map in $E^*(-)$. Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

When $E = H\mathbb{F}_p$, this injective class leads to the classical Adams spectral sequence.

We always assume that our projective and injective classes are **stable**, that is, that they are closed under suspension and desuspension.

Adams resolutions

Definition. An **Adams resolution** of an object Y in \mathcal{T} with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram

$$\begin{array}{ccccccc} Y = Y_0 & \xleftarrow{i_0} & Y_1 & \xleftarrow{i_1} & Y_2 & \xleftarrow{i_2} & Y_3 \xleftarrow{\quad} \dots \\ & \searrow p_0 & \nearrow \delta_0 & \searrow p_1 & \nearrow \delta_1 & \searrow p_2 & \nearrow \delta_2 \\ & & I_0 & & I_1 & & I_2 \quad \dots \end{array}$$

where each I_s is injective, each map i_s is in \mathcal{N} , and the triangles are triangles.

Axiom (iii) says exactly that you can form such a resolution.

Adams resolutions biject with **injective resolutions**, which are diagrams

$$0 \longrightarrow Y \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \dots$$

that give exact sequences under $\mathcal{T}(-, I)$ for each I in \mathcal{I} .

Given objects X and Y and an Adams resolution

$$\begin{array}{ccccccc}
 Y = Y_0 & \xleftarrow{i_0} & Y_1 & \xleftarrow{i_1} & Y_2 & \xleftarrow{i_2} & Y_3 \leftarrow \dots \\
 & \searrow p_0 & \nearrow \delta_0 & \searrow p_1 & \nearrow \delta_1 & \searrow p_2 & \nearrow \delta_2 \\
 & & I_0 & \xrightarrow{d_1} & I_1 & \xrightarrow{d_1} & I_2 \quad \dots
 \end{array}$$

of Y , applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence; it is called the **Adams spectral sequence**.

The E_1 term is $E_1^{s,t} = \mathcal{T}(\Sigma^{t-s} X, I_s)$, and the first differential d_1 is given by composition with

$$d_1 := p\delta : I_s \twoheadrightarrow Y_{s+1} \longrightarrow I_{s+1}.$$

The E_2 term is $\text{Ext}_{\mathcal{I}}^s(\Sigma^t X, Y)$, essentially by definition.

d_1 is a primary operation

The \mathcal{I} -cohomology of an object X is the family of abelian groups $H^I(X) := \mathcal{T}(X, I)$ indexed by the injective objects $I \in \mathcal{I}$.

A **primary operation** in \mathcal{I} -cohomology is a natural transformation $H^I(X) \rightarrow H^J(X)$ of functors $\mathcal{T} \rightarrow \text{Ab}$. Equivalently, it is a map $I \rightarrow J$ in \mathcal{I} .

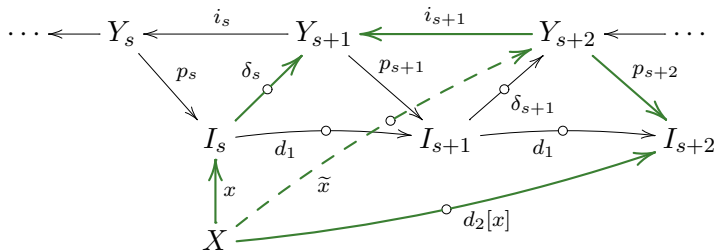
Clearly, $d_1 : I_s \rightarrow \Sigma I_{s+1}$ is a primary operation.

Our goal is to describe the **higher** differentials using **higher** operations.

Adams d_2 differential

Recall that E_2 is the homology of $\mathcal{T}(X, I_s)$ w.r.t. d_1 .

Given a class $[x]$ in the E_2 term of an Adams spectral sequence, $d_2[x]$ is defined in the following way:



$d_2[x]$ is a subset of $\mathcal{T}(X, \Sigma I_{s+2})$ (a coset of the image of d_1).

We'll describe this subset using "higher operations".

Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ be a diagram in \mathcal{T} .

The **Toda bracket** $\langle f_3, f_2, f_1 \rangle \subseteq \mathcal{T}(\Sigma X_0, X_3)$ consists of all composites $\beta \circ \Sigma\alpha: \Sigma X_0 \rightarrow X_3$, where α and β appear in a commutative diagram

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{f_1} & X_1 & & \Sigma X_0 & \xrightarrow{-\Sigma f_1} & \Sigma X_1 \\
 \alpha \downarrow & & \parallel & & \Sigma\alpha \downarrow & & \parallel \\
 \Sigma^{-1}C_{f_2} & \xrightarrow{j} & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & C_{f_2} & \xrightarrow{-\Sigma j} & \Sigma X_1 \\
 & & \parallel & & \parallel & & \downarrow \beta & & \\
 & & X_2 & \xrightarrow{f_3} & X_3 & & & &
 \end{array}$$

where the middle row is a triangle.

Rotating the middle triangle introduces a sign.

Instead of a triangle involving f_2 , one can make an equivalent definition using a triangle based on f_1 :

Proposition. The Toda bracket $\langle f_3, f_2, f_1 \rangle$ consists of all maps $\psi: \Sigma X_0 \rightarrow X_3$ that appear in a commutative diagram

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{f_1} & X_1 & \longrightarrow & C_{f_1} & \longrightarrow & \Sigma X_0 \\
 \parallel & & \parallel & & \downarrow & & \downarrow \psi \\
 X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3,
 \end{array}$$

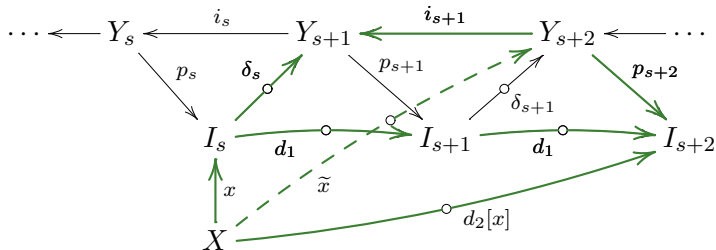
where the top row is a triangle.

There is also an equivalent dual definition involving f_3 .

The **indeterminacy** can be explicitly described.

Adams d_2 in terms of Toda brackets

Recall our Adams resolution and the definition of d_2 :



Proposition (“Known to the experts”). $d_2[x] \subseteq \langle d_1, d_1, x \rangle$.

Note. The inclusion can be proper.

Adams d_2 in terms of Toda brackets, II

The inclusion $d_2[x] \subseteq \langle d_1, d_1, x \rangle$ can be made sharper.

Proposition (C-Frankland). $d_2[x] = \langle d_1, p_{s+1}, \delta_s x \rangle = \langle d_1, d_1, x \rangle^\beta$.

The first equality is an elementary exercise, using the properties of injective classes. The second requires some explanation:

Recall that $\langle f_3, f_2, f_1 \rangle$ was defined to consist of certain composites

$$\Sigma X_0 \xrightarrow{\Sigma \alpha} C_{f_2} \xrightarrow{\beta} X_3.$$

The notation $\langle f_3, f_2, f_1 \rangle^\beta$ denotes the subset of the Toda bracket with β held **fixed** and only α allowed to vary.

Adams d_2 in terms of Toda brackets, III

From the definition $d_1 = p_{s+1}\delta_s$ and the octahedral axiom, we get

$$\begin{array}{ccccccc}
 & & \Sigma Y_{s+2} & \equiv & \Sigma Y_{s+2} & & \\
 & & \downarrow \Sigma i_{s+1} & & \downarrow i_s(\Sigma i_{s+1}) & & \\
 I_s & \xrightarrow{\delta_s} & \Sigma Y_{s+1} & \xrightarrow{i_s} & \Sigma Y_s & \xrightarrow{\Sigma p_s} & \Sigma I_s \\
 \parallel & & \downarrow \Sigma p_{s+1} & & \vdots & & \parallel \\
 I_s & \xrightarrow{d_1} & \Sigma I_{s+1} & \dashrightarrow & W & \dashrightarrow & \Sigma I_s \\
 & & \downarrow \Sigma \delta_{s+1} & & \vdots & & \\
 & & \Sigma^2 Y_{s+2} & \equiv & \Sigma^2 Y_{s+2} & & \\
 & \swarrow \Sigma d_1 & & & \swarrow \beta & & \\
 & & \Sigma^2 p_{s+2} & & & & \\
 & & \swarrow & & \swarrow & & \\
 \Sigma^2 I_{s+2} & & & & & &
 \end{array}$$

with all rows and columns triangles. Define β as shown. Ponder.

Definition. Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$, define the **Toda family** $T(f_3, f_2, f_1)$ to consist of all pairs $(\beta, \Sigma\alpha)$, where α and β appear in a commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma X_0 & \xrightarrow{\quad} & \Sigma X_1 \\
 & & & & \downarrow \Sigma\alpha & \searrow -\Sigma f_1 & \parallel \\
 X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{\quad} & C_{f_2} & \xrightarrow{\quad} & \Sigma X_1 \\
 & & \parallel & & \downarrow \beta & & \\
 & & X_2 & \xrightarrow{f_3} & X_3, & &
 \end{array}$$

with middle row a triangle.

Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} X_n$, define the **Toda bracket** $\langle f_n, \dots, f_1 \rangle \subseteq \mathcal{T}(\Sigma^{n-2}X_0, X_n)$ inductively as follows:

If $n = 2$, it is the set consisting of just the composite $f_2 f_1$.

If $n > 2$, it is the union of the sets $\langle \beta, \Sigma\alpha, \Sigma f_{n-3}, \dots, \Sigma f_1 \rangle$, where $(\beta, \Sigma\alpha)$ is in $T(f_n, f_{n-1}, f_{n-2})$.

4-fold Toda bracket

Example. We have

$$\langle f_4, f_3, f_2, f_1 \rangle = \bigcup_{\beta, \alpha} \langle \beta, \Sigma\alpha, \Sigma f_1 \rangle = \bigcup_{\beta, \alpha} \bigcup_{\beta', \alpha'} \{ \beta' \circ \Sigma\alpha' \}.$$

$$\begin{array}{ccc}
 \Sigma^2 X_0 \xrightarrow{\Sigma\alpha'} C_{\Sigma\alpha} \longrightarrow \Sigma^2 X_1 & & \text{row} = -\Sigma^2 f_1 \\
 \uparrow \beta' & \searrow & \\
 \Sigma X_1 \xrightarrow{\Sigma\alpha} C_{f_3} \longrightarrow \Sigma X_2 & & \text{row} = -\Sigma f_2 \\
 \uparrow \beta & \searrow & \\
 X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} X_4 & & \\
 \uparrow 0 & &
 \end{array}$$

Cohen, Shipley and Sagave gave a definition of Toda brackets using **filtered objects**, and this shows that McKeown's definition agrees with theirs.

Self-duality for higher Toda brackets

The definition is asymmetrical. What happens in the opposite category?

More generally, we can reduce an n -fold Toda bracket to a 2-fold Toda bracket in $(n - 2)!$ ways, inserting the Toda family operation in any position.

Lemma (C-Frankland). The pair $(\beta, \Sigma\alpha)$ is in $T(T(f_4, f_3, f_2), \Sigma f_1)$ iff the pair $(-\beta, \Sigma\alpha)$ is in $T(f_4, T(f_3, f_2, f_1))$.

This is stronger than saying that the two ways of computing the Toda bracket $\langle f_4, f_3, f_2, f_1 \rangle$ are negatives, and the stronger statement will be important for us.

The proof is a careful application of the octahedral axiom.

Self-duality, II

For j_1, j_2, \dots, j_{n-2} with $0 \leq j_i < i$, write

$$T_{j_1}(T_{j_2}(T_{j_3}(\cdots T_{j_{n-2}}(f_n, \dots, f_1) \cdots)))$$

for the subset obtained by applying T in the spot with j_{n-2} maps to the left, then applying T in the spot with j_{n-3} maps to the left, etc.

Our original definition corresponds to $T_0(T_0(\cdots T_0(f_n, \dots, f_1) \cdots))$.

Theorem (C-Frankland). If you compute the Toda bracket using the sequence j_1, j_2, \dots, j_{n-2} , it equals the original Toda bracket up to the sign $(-1)^{\sum j_i}$.

Proof. One can give an inductive argument showing that the Lemma lets you convert any such sequence into any other, using the “move” $j, j \longleftrightarrow j, j + 1$. The move changes the sign and the parity of the sum. □

Corollary. The higher Toda brackets are self-dual up to sign.

Adams d_r in terms of Toda brackets

Recall:

Proposition (C-Frankland). $d_2[x] = \langle d_1, p_{s+1}, \delta_s x \rangle = \langle d_1, d_1, x \rangle^\beta$.

Theorem (C-Frankland). d_r can be expressed in terms of $(r + 1)$ -fold Toda brackets as:

$$d_r[x] = \langle d_1, d_1, \dots, d_1, p_{s+1}, \delta_s x \rangle = \langle d_1, d_1, \dots, d_1, x \rangle_{\text{fixed}}$$

The first equality is straightforward, using the dual Toda bracket.

In the second equality, “fixed” means that you choose a particular filtered object derived from the Adams resolution, which fixes all of the choices except the very last β .

(Details are in our paper [1].)

Sparse rings

Fix a compact object R of \mathcal{T} and write $\pi_n^R(X)$ for $\mathcal{T}(\Sigma^n R, X)$.

We consider the Adams SS for the projective class generated by R abutting to $\mathcal{T}(X, Y)$.

Theorem (C-Frankland). If there exists an $N > 0$ such that $\pi_*^R R$ and $\pi_*^R X$ are concentrated in degrees congruent to $0 \pmod N$, then for any class y in $E_r^{s,t}$ with $r \leq N$, we have

$$d_r[y] = \langle y, d_1, d_1, \dots, d_1 \rangle.$$

Example. Let R be a ring spectrum with $\pi_*(R)$ concentrated in degrees congruent to $0 \pmod N$. Let \mathcal{T} be the homotopy category of R -modules. Then the Adams SS is the universal coefficient SS

$$\mathrm{Ext}_{\pi_*^R}^s(\Sigma^t \pi_* X, \pi_* Y) \Rightarrow h\mathrm{Mod}_R(\Sigma^{t-s} X, Y).$$

E.g. the Johnson-Wilson spectrum $E(n)$ satisfies this for $N = 2(p-1)$.

3-fold Toda brackets determine the triangulation

I'd like to end by advertising this nice result due to Heller (1968) with a cleaner formulation and proof due to Muro (2006 slides, 2015 e-mail):

Theorem. The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle iff

(i) the sequence of abelian groups

$$\mathcal{T}(A, \Sigma^{-1}Z) \rightarrow \mathcal{T}(A, X) \rightarrow \mathcal{T}(A, Y) \rightarrow \mathcal{T}(A, Z) \rightarrow \mathcal{T}(A, \Sigma X)$$

is exact for every object A of \mathcal{T} , and

(ii) the Toda bracket $\langle h, g, f \rangle \subseteq \mathcal{T}(\Sigma X, \Sigma X)$ contains the identity map $1_{\Sigma X}$.

The proof is essentially the Yoneda Lemma and the Five Lemma.

3-fold Toda brackets determine the higher ones

Corollary. Given the suspension functor $\Sigma: \mathcal{T} \rightarrow \mathcal{T}$, 3-fold Toda brackets in \mathcal{T} determine the triangulated structure. In particular, 3-fold Toda brackets determine the higher Toda brackets, via the triangulation.

Remark. It is unclear to us if the higher Toda brackets can be expressed directly in terms of 3-fold brackets.

Details are in [1], and these slides are on my website.

- [1] J.D. Christensen and M. Frankland. Higher Toda brackets and the Adams spectra sequence in triangulated categories. Algebraic & Geometric Topology **17**(5) (2017), 2687–2735. arxiv:1510.09216.

Thanks for listening!