Higher Toda brackets and the Adams spectral sequence

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Outline:

- Triangulated categories and injective classes
- The Adams spectral sequence
- 3-fold Toda brackets, and the relation to d_2
- Higher Toda brackets, and the relation to d_r

Triangulated categories

A triangulated category is an additive category $\mathcal{T}$ equipped with an equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$, and with a specified collection of triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X. \tag{1}$$

These must satisfy the following axioms motivated by (co)fibre sequences in topology.

TR0: The triangles are closed under isomorphism.
The following is a triangle:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X.$$

TR1: Every map $X \to Y$ is part of a triangle (1).

TR2: (1) is a triangle iff (2) is a triangle:

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y. \tag{2}$$

Triangulated categories, II

$\mathcal{T}$ additive, $\Sigma : \mathcal{T} \to \mathcal{T}$ an equivalence.

TR0: Triangles are closed under isomorphism and contain the trivial triangle.

TR1: Every map appears in a triangle.

TR2: Triangles can be rotated.

TR3: Given a solid diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle u}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle \Sigma u} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

in which the rows are triangles, the dotted fill-in exists making the two squares commute.

TR4: The octahedral axiom holds.

Examples and consequences

Example. The homotopy category of spectra.

Example. The derived category of a ring.

Example. The stable module category of a group algebra.

Example. The homotopy category of any stable Quillen model category.

Consequences: (1) For any object A, the sequences

$$\cdots \longrightarrow \mathcal{T}(A,X) \longrightarrow \mathcal{T}(A,Y) \longrightarrow \mathcal{T}(A,Z) \longrightarrow \mathcal{T}(A,\Sigma X) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow \mathcal{T}(X,A) \longleftarrow \mathcal{T}(Y,A) \longleftarrow \mathcal{T}(Z,A) \longleftarrow \mathcal{T}(\Sigma X,A) \longleftarrow \cdots$$

are exact sequences of abelian groups.

(2) The triangle containing a map $X \to Y$ is unique up to (non-unique) isomorphism.

Injective classes

Eilenberg and Moore (1965) gave a framework for homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following:

Definition. An injective class in $\mathcal{T}$ is a pair $(\mathcal{I}, \mathcal{N})$, where $\mathcal{I} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{I}$ consists of exactly the objects I such that every composite $X \to Y \to I$ is zero for each $X \to Y$ in $\mathcal{N}$,

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $X \to Y \to I$ is zero for each I in $\mathcal{I}$,

(iii) for each Y in $\mathcal{T}$, there is a triangle $X \to Y \to I$ with I in $\mathcal{I}$ and $X \to Y$ in $\mathcal{N}$.

The first two conditions are easy to satisfy. The third says that there are enough injectives.

Examples of injective classes

Example. Let E be an object in any triangulated category $\mathcal{T}$ with infinite products. Take $\mathcal{I}$ to be all retracts of products of suspensions of E and $\mathcal{N}$ to consist of all maps $X \to Y$ such that every composite $X \to Y \to I$ is zero, for I in $\mathcal{I}$. Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

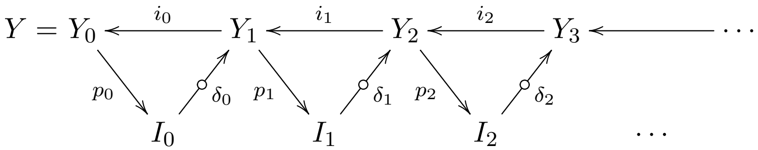
If we write $E^k(-)$ for the cohomological representable functor $\mathcal{T}(-, \Sigma^k E)$, then $\mathcal{N}$ consists of the maps inducing the zero map under $E^*(-)$.

Example. In the category of spectra, if we take $E = H\mathbb{F}_p$, this injective class leads to the classical Adams spectral sequence.

We always assume that our injective classes are stable, that is, that they are closed under suspension and desuspension.

Adams resolutions

Definition. An Adams resolution of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram



where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

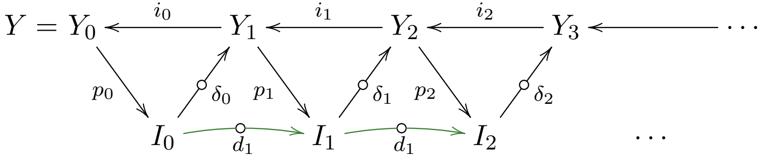
Axiom (iii) says exactly that you can form such a resolution.

Adams resolutions biject with injective resolutions with respect to the injective class.

The Adams spectral sequence

Miller, 1974; C, 1997

Given objects X and Y and an Adams resolution



of Y, applying $\mathcal{T}(X,-)$ leads to an exact couple and therefore a spectral sequence; it is called the Adams spectral sequence.

The E_1 term is $E_1^{s,t} = \mathcal{T}(\Sigma^{t-s}X, I_s)$, and the first differential d_1 is given by composition with

$$d_1 := p\delta : I_s \rightarrow Y_{s+1} \longrightarrow I_{s+1}.$$

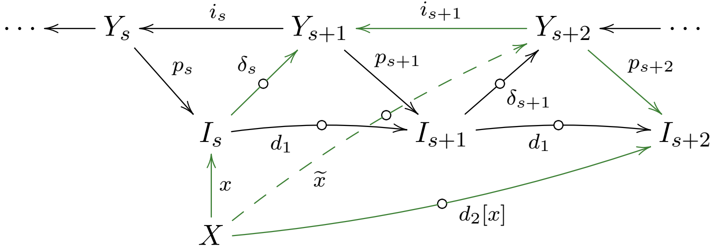
The E_2 term is $\mathrm{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$, essentially by definition.

We regard d_1 as a primary operation.

Adams d_2 differential

Recall that E_2 is the homology of $\mathcal{T}(X, I_s)$ w.r.t. d_1.

Given a class $[x]$ in the E_2 term of an Adams spectral sequence, $d_2[x]$ is defined in the following way:



$d_2[x]$ is a subset of $\mathcal{T}(X, I_{s+2})$. We'll describe this subset using "higher operations".

3-fold Toda brackets

Toda, 1962

Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ be a diagram in $\mathcal{T}$.

The Toda bracket $\langle f_3, f_2, f_1 \rangle \subseteq \mathcal{T}(\Sigma X_0, X_3)$ consists of all composites $\beta \circ \Sigma\alpha \colon \Sigma X_0 \to X_3$, where α and β appear in a commutative diagram

$$\begin{array}{ccccccc}
 & & X_0 & \xrightarrow{f_1} & X_1 & & & & \\
 & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\
 & & \Sigma^{-1}C_{f_2} & \longrightarrow & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & C_{f_2} \\
 & & & & & & \| & & \downarrow{\scriptstyle \beta} \\
 & & & & & & X_2 & \xrightarrow{f_3} & X_3,
\end{array}$$

where the middle row is a triangle.

The indeterminacy can be explicitly described, and there are other equivalent definitions.

Adams d_2 in terms of Toda brackets

Proposition (C-Frankland). $d_2[x] = \langle d_1, p_{s+1}, \delta_s x \rangle = \langle d_1 \overset{\beta}{,} d_1, x \rangle$.

The first equality is an elementary exercise, using the properties of injective classes. The second requires some explanation.

Recall that $\langle f_3, f_2, f_1 \rangle$ was defined to consist of certain composites

$$\Sigma X_0 \xrightarrow{\Sigma\alpha} C_{f_2} \xrightarrow{\beta} X_3.$$

The notation $\langle f_3 \overset{\beta}{,} f_2, f_1 \rangle$ denotes the subset of the Toda bracket with β held fixed and only α allowed to vary.

The choice of β is determined from the Adams resolution and the octahedral axiom.

Adams d_r in terms of Toda brackets

Following Cohen, Shipley and McKeown, we define r-fold Toda brackets in any triangulated category, and prove basic properties about them. Our main result is:

Theorem (C-Frankland)**.** d_r can be expressed in terms of $(r+1)$-fold Toda brackets as:

$$d_r[x] = \langle d_1, d_1, \ldots, d_1, p_{s+1}, \delta_s x \rangle = \langle d_1, d_1, \ldots, d_1, x \rangle_{\text{fixed}}$$

The first equality is straightforward, using our results.

In the second equality, "fixed" means that you choose a particular "filtered object" derived from the Adams resolution, which fixes all of the choices except the very last α.

Details are in arxiv:1510.09216, and these slides are on my website.

Thanks for listening!

Overflow slides

The remaining slides are just in case I have extra time.

Higher Toda brackets

McKeown, nLab, 2012

Definition. Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$, define the Toda family $\mathrm{T}(f_3, f_2, f_1)$ to consist of all pairs $(\beta, \Sigma\alpha)$, where α and β appear in a commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma X_0 & \xrightarrow{-\Sigma f_1} & \Sigma X_1 \\
 & & & & \downarrow{\scriptstyle \Sigma\alpha} & & \| \\
X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & C_{f_2} & \longrightarrow & \Sigma X_1 \\
 & & \| & & \downarrow{\scriptstyle \beta} & & \\
 & & X_2 & \xrightarrow{f_3} & X_3, & &
\end{array}$$

with middle row a triangle.

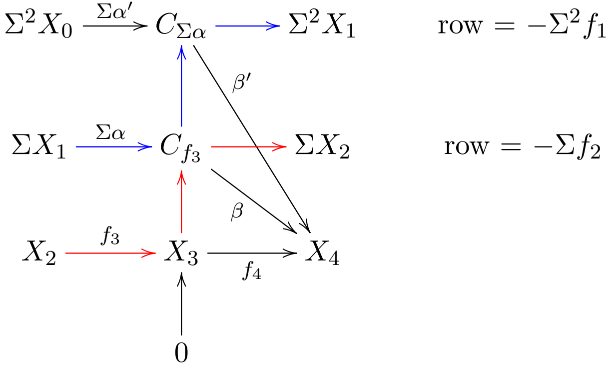
Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n$, define the Toda bracket $\langle f_n, \ldots, f_1 \rangle \subseteq \mathcal{T}(\Sigma^{n-2} X_0, X_n)$ inductively as follows:

If $n = 2$, it is the set consisting of just the composite $f_2 f_1$.

If $n > 2$, it is the union of the sets $\langle \beta, \Sigma\alpha, \Sigma f_{n-3}, \ldots, \Sigma f_1 \rangle$, where $(\beta, \Sigma\alpha)$ is in $T(f_n, f_{n-1}, f_{n-2})$.

4-fold Toda bracket

Example. We have
$\langle f_4, f_3, f_2, f_1\rangle = \bigcup_{\beta,\alpha} \langle \beta, \Sigma\alpha, \Sigma f_1\rangle = \bigcup_{\beta,\alpha} \bigcup_{\beta',\alpha'} \{\beta' \circ \Sigma\alpha'\}$.



The middle column is what is called a filtered object by Cohen, Shipley and Sagave, and so this reproduces their definition.

Self-duality for higher Toda brackets

The definition is asymmetrical. What happens in the opposite category?

More generally, we can reduce an n-fold Toda bracket to a 2-fold Toda bracket in $(n-2)!$ ways, inserting the Toda family operation in any position.

Lemma (C-Frankland)**.** The pair $(\beta, \Sigma\alpha)$ is in $T(T(f_4, f_3, f_2), \Sigma f_1)$ iff the pair $(-\beta, \Sigma\alpha)$ is in $T(f_4, T(f_3, f_2, f_1))$.

This is stronger than saying that the two ways of computing the Toda bracket $\langle f_4, f_3, f_2, f_1 \rangle$ are negatives, and the stronger statement will be important for us.

The proof is a careful application of the octahedral axiom.

Self-duality, II

For $j_1, j_2, \ldots, j_{n-2}$ with $0 \leq j_i < i$, write

$$T_{j_1}(T_{j_2}(T_{j_3}(\cdots T_{j_{n-2}}(f_n, \ldots, f_1)\cdots)))$$

for the subset obtained by applying T in the spot with j_{n-2} maps to the left, then applying T in the spot with j_{n-1} maps to the left, etc.

Our original definition corresponds to $T_0(T_0(\cdots T_0(f_n, \ldots, f_1)\cdots))$.

Theorem (C-Frankland)**.** If you compute the Toda bracket using the sequence $j_1, j_2, \ldots, j_{n-2}$, it equals the original Toda bracket up to the sign $(-1)^{\sum j_i}$.

Proof. One can give an inductive argument showing that the Lemma lets you convert any such sequence into any other, using the "move" $j, j \longleftrightarrow j, j+1$. Animation: `http://turl.ca/todaanim`
The move changes the sign and the parity of the sum. □

Corollary. The higher Toda brackets are self-dual up to sign.

An example in the stable module category

Let $R = kC_4 = k[x]/x^4$ with char $k = 2$.
Let $M = R/x^2$. In $\mathrm{StMod}(R)$, $\Omega M = M$.
With respect to the projective class generated by k,

$$
\begin{array}{ccccccccccc}
M & & \xrightarrow{\ \mu_x\ } & & M & & \xrightarrow{\ \mu_x\ } & & M & \xrightarrow{\ \mu_x\ } & M \longrightarrow \cdots \\
& \nwarrow^{p} & & \swarrow_{\delta} & & \nwarrow^{p} & & \swarrow_{\delta} & & \nwarrow^{p} \quad \swarrow_{\delta} & \\
& & k \oplus \Omega k & & & & k \oplus \Omega k & & & k \oplus \Omega k, &
\end{array}
$$

is an Adams resolution of M, for certain p and δ.

Given any non-zero map $\kappa : k \oplus \Omega k \to M$, one can show that $d_2[\kappa]$ has no indeterminacy, while $\langle \kappa, d_1, d_1 \rangle$ has non-trivial indeterminacy, so the containment

$$d_2[\kappa] = \langle \kappa, d_1 \overset{\beta}{,} d_1 \rangle \subseteq \langle \kappa, d_1, d_1 \rangle$$

is proper.

3-fold Toda brackets determine the triangulation

I'd like to end by advertising this nice result due to Heller (1968) with a cleaner formulation and proof due to Muro (2006 slides, 2015 e-mail):

Theorem. The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle iff

(i) the sequence of abelian groups

$$\mathcal{T}(A, \Sigma^{-1} Z) \xrightarrow{(\Sigma^{-1} h)_*} \mathcal{T}(A, X) \xrightarrow{f_*} \mathcal{T}(A, Y) \xrightarrow{g_*} \mathcal{T}(A, Z) \xrightarrow{h_*} \mathcal{T}(A, \Sigma X)$$

is exact for every object A of $\mathcal{T}$, and

(ii) the Toda bracket $\langle h, g, f \rangle \subseteq \mathcal{T}(\Sigma X, \Sigma X)$ contains the identity map $1_{\Sigma X}$.

The proof is essentially the Yoneda Lemma and the Five Lemma.

3-fold Toda brackets determine the higher ones

Corollary. Given the suspension functor $\Sigma\colon \mathcal{T} \to \mathcal{T}$, 3-fold Toda brackets in $\mathcal{T}$ determine the triangulated structure. In particular, 3-fold Toda brackets determine the higher Toda brackets, via the triangulation.

Remark. It is unclear to us if the higher Toda brackets can be expressed directly in terms of 3-fold brackets.

Thanks for listening!

These slides are available on my website.