Higher Toda brackets and the Adams spectral sequence

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Outline:

- The Adams spectral sequence
- 3-fold Toda brackets, and the relation to d_2
- Higher Toda brackets, and the relation to d_r
- 3-fold Toda brackets determine the rest

Triangulated categories

A triangulated category is an additive category $\mathcal{T}$ equipped with a functor $\Sigma : \mathcal{T} \to \mathcal{T}$ that is an equivalence, and with a specified collection of triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X. \tag{1}$$

These must satisfy the following axioms motivated by (co)fibre sequences in topology.

TR0: The triangles are closed under isomorphism. The following is a triangle:

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X.$$

TR1: Every map $X \to Y$ is part of a triangle (1).

TR2: (1) is a triangle iff (2) is a triangle:

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y. \tag{2}$$

Triangulated categories, II

$\mathcal{T}$ additive, $\Sigma : \mathcal{T} \to \mathcal{T}$ an equivalence.

TR0: Triangles are closed under isomorphism and contain the trivial triangle.

TR1: Every map appears in a triangle.

TR2: Triangles can be rotated.

TR3: Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

in which the rows are triangles, the dotted fill-in exists making the two squares commute.

TR4: The octahedral axiom holds. (Some details later.)

Examples and consequences

Example. The homotopy category of spectra.

Example. The derived category of a ring.

Example. The stable module category of a group algebra.

Consequences: (1) For any object A, the sequences

$$\cdots \longrightarrow \mathcal{T}(A,X) \longrightarrow \mathcal{T}(A,Y) \longrightarrow \mathcal{T}(A,Z) \longrightarrow \mathcal{T}(A,\Sigma X) \longrightarrow \cdots$$

and

$$\cdots \longleftarrow \mathcal{T}(X,A) \longleftarrow \mathcal{T}(Y,A) \longleftarrow \mathcal{T}(Z,A) \longleftarrow \mathcal{T}(\Sigma X,A) \longleftarrow \cdots$$

are exact.

(2) The triangle containing a map $X \to Y$ is unique up to (non-unique) isomorphism.

Projective and injective classes

Eilenberg and Moore (1965) gave a framework for homological algebra in any pointed category. When the category is triangulated, their axioms are equivalent to the following:

Definition. A projective class in $\mathcal{T}$ is a pair $(\mathcal{P}, \mathcal{N})$, where $\mathcal{P} \subseteq \operatorname{ob} \mathcal{T}$ and $\mathcal{N} \subseteq \operatorname{mor} \mathcal{T}$, such that:

(i) $\mathcal{P}$ consists of exactly the objects P such that every composite $P \to X \to Y$ is zero for each $X \to Y$ in $\mathcal{N}$,

(ii) $\mathcal{N}$ consists of exactly the maps $X \to Y$ such that every composite $P \to X \to Y$ is zero for each P in $\mathcal{P}$,

(iii) for each X in $\mathcal{T}$, there is a triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{N}$.

The first two conditions are easy to satisfy. The third says that there are enough projectives.

An injective class in $\mathcal{T}$ is a projective class in $\mathcal{T}^{\mathrm{op}}$.

Examples of projective and injective classes

Example. In spectra, take $\mathcal{P}$ to be all retracts of wedges of spheres and $\mathcal{N}$ to consist of all maps inducing the zero map in homotopy groups. Then $(\mathcal{P}, \mathcal{N})$ is a projective class.

The analogous construction works starting with any set of objects in any triangulated category with all coproducts.

Example. Dually, if E is any spectrum, take $\mathcal{I}$ to be all retracts of products of suspensions of E and $\mathcal{N}$ to consist of all maps inducing the zero map in $E^*(-)$. Then $(\mathcal{I}, \mathcal{N})$ is an injective class.

When $E = H\mathbb{F}_p$, this injective class leads to the classical Adams spectral sequence.

We always assume that are projective and injective classes are stable, that is, that they are closed under suspension and desuspension.

Adams resolutions

Definition. An Adams resolution of an object Y in $\mathcal{T}$ with respect to an injective class $(\mathcal{I}, \mathcal{N})$ is a diagram

$$\begin{tikzcd}
Y = Y_0 \arrow[dr, "p_0"'] & & Y_1 \arrow[ll, "i_0"'] \arrow[dr, "p_1"'] & & Y_2 \arrow[ll, "i_1"'] \arrow[dr, "p_2"'] & & Y_3 \arrow[ll, "i_2"'] & & \cdots \arrow[ll] \\
 & I_0 \arrow[ur, "\delta_0"'] & & I_1 \arrow[ur, "\delta_1"'] & & I_2 \arrow[ur, "\delta_2"'] & & \cdots &
\end{tikzcd}$$

where each I_s is injective, each map i_s is in $\mathcal{N}$, and the triangles are triangles.

Axiom (iii) says exactly that you can form such a resolution.

Adams resolutions biject with injective resolutions, which are diagrams

$$0 \longrightarrow Y \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

that give exact sequences under $\mathcal{T}(-, I)$ for each I in $\mathcal{I}$.

The Adams spectral sequence

Miller, 1974; C, 1997

Given objects X and Y and an Adams resolution

$$\begin{tikzcd}
Y = Y_0 \arrow[dr, "p_0"'] & & Y_1 \arrow[ll, "i_0"'] \arrow[dr, "p_1"'] & & Y_2 \arrow[ll, "i_1"'] \arrow[dr, "p_2"'] & & Y_3 \arrow[ll, "i_2"'] & & \cdots \arrow[ll] \\
& I_0 \arrow[ur, "\delta_0"'] & & I_1 \arrow[ur, "\delta_1"'] & & I_2 \arrow[ur, "\delta_2"'] & & \cdots &
\end{tikzcd}$$

of Y, applying $\mathcal{T}(X, -)$ leads to an exact couple and therefore a spectral sequence; it is called the Adams spectral sequence.

The E_1 term is $E_1^{s,t} = \mathcal{T}(\Sigma^{t-s} X, I_s)$, and the first differential d_1 is given by composition with

$$d_1 := p\delta : I_s \overset{\circ}{\longrightarrow} Y_{s+1} \longrightarrow I_{s+1}.$$

Proposition. The E_2 term is given by $\mathrm{Ext}^s_{\mathcal{I}}(\Sigma^t X, Y)$.

d_1 is a primary operation

The $\mathcal{I}$-cohomology of an object X is the family of abelian groups $H^I(X) := \mathcal{T}(X, I)$ indexed by the injective objects $I \in \mathcal{I}$.

A primary operation in $\mathcal{I}$-cohomology is a natural transformation $H^I(X) \to H^J(X)$ of functors $\mathcal{T} \to \mathrm{Ab}$. Equivalently, it is a map $I \to J$ in $\mathcal{I}$.

Clearly, $d_1 : I_s \to \Sigma I_{s+1}$ is a primary operation.

Our goal is to describe the higher differentials using higher operations.

3-fold Toda brackets

Let $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$ be a diagram in $\mathcal{T}$.

The Toda bracket $\langle f_3, f_2, f_1 \rangle \subseteq \mathcal{T}(\Sigma X_0, X_3)$ consists of all composites $\beta \circ \Sigma\alpha \colon \Sigma X_0 \to X_3$, where α and β appear in a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, "\alpha"'] & X_1 \arrow[d, equal] & & \Sigma X_0 \arrow[r, "-\Sigma f_1"] \arrow[d, "\Sigma\alpha"'] & \Sigma X_1 \arrow[d, equal] \\
\Sigma^{-1} C_{f_2} \arrow[r, "j"] & X_1 \arrow[r, "f_2"] & X_2 \arrow[r] \arrow[d, equal] & C_{f_2} \arrow[r, "-\Sigma j"] \arrow[d, "\beta"] & \Sigma X_1 \\
& & X_2 \arrow[r, "f_3"] & X_3,
\end{tikzcd}$$

where the middle row is a triangle.

Rotating the middle triangle introduces a sign.

3-fold Toda brackets

Toda, 1962; Sagave, 2008

Instead of a triangle involving f_2, one can make an equivalent definition using a triangle based on f_1:

Proposition. The Toda bracket $\langle f_3, f_2, f_1 \rangle$ consists of all maps $\psi \colon \Sigma X_0 \to X_3$ that appear in a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & C_{f_1} \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d, "\psi"] \\
X_0 \arrow[r, "f_1"] & X_1 \arrow[r, "f_2"] & X_2 \arrow[r, "f_3"] & X_3,
\end{tikzcd}$$

where the top row is a triangle.

There is also an equivalent dual definition involving f_3.

The indeterminacy can be described explicitly.

Adams d_2 in terms of Toda brackets

Given a class $[x]$ in the E_2 term of an Adams spectral sequence, $d_2[x]$ is computed as shown:

$$
\begin{tikzcd}
\cdots & Y_s \arrow[l] \arrow[dr, "p_s"] & & Y_{s+1} \arrow[ll, "i_s"'] \arrow[dr, "p_{s+1}"] & & Y_{s+2} \arrow[ll, "i_{s+1}"'] \arrow[dr, "p_{s+2}"] & & \cdots \arrow[ll] \\
& & I_s \arrow[ur, "\delta_s"] \arrow[rr, "d_1"'] & & I_{s+1} \arrow[ur, "\delta_{s+1}"'] \arrow[rr, "d_1"'] & & I_{s+2} & \\
& & X \arrow[u, "x"'] \arrow[uurrr, dashed, "\widetilde{x}"'] \arrow[urrrr, bend right, "d_2[x]"'] & & & & &
\end{tikzcd}
$$

Proposition ("Known to the experts"). $d_2[x] \subseteq \langle d_1, d_1, x \rangle$.

Note. The inclusion can be proper, and I'll illustrate this later if there is time.

Adams d_2 in terms of Toda brackets, II

The inclusion $d_2[x] \subseteq \langle d_1, d_1, x \rangle$ can be made sharper.

Proposition (C-Frankland). $d_2[x] = \langle d_1, p_{s+1}, \delta_s x \rangle = \langle d_1 \overset{\beta}{,} d_1, x \rangle$.

The first equality is an elementary exercise, using the properties of injective classes. The second requires some explanation. Recall that $\langle f_3, f_2, f_1 \rangle$ was defined to consist of certain composites

$$\Sigma X_0 \xrightarrow{\Sigma\alpha} C_{f_2} \xrightarrow{\beta} X_3.$$

The notation $\langle f_3 \overset{\beta}{,} f_2, f_1 \rangle$ denotes the subset of the Toda bracket with β held fixed and only α allowed to vary.

Adams d_2 in terms of Toda brackets, III

From the definition $d_1 = p_{s+1}\delta_s$ and the octahedral axiom, we get

$$
\begin{tikzcd}
 & \Sigma Y_{s+2} \arrow[r, equal] \arrow[d, "\Sigma i_{s+1}"'] & \Sigma Y_{s+2} \arrow[d, "i_s(\Sigma i_{s+1})"] & \\
I_s \arrow[r, "\delta_s"] \arrow[d, equal] & \Sigma Y_{s+1} \arrow[r, "i_s"] \arrow[d, "\Sigma p_{s+1}"'] & \Sigma Y_s \arrow[r, "\Sigma p_s"] \arrow[d, dashed] & \Sigma I_s \arrow[d, equal] \\
I_s \arrow[r, "d_1"] & \Sigma I_{s+1} \arrow[r, dashed] \arrow[d, "\Sigma \delta_{s+1}"] \arrow[ddl, bend right, "\Sigma d_1"'] & W \arrow[r, dashed] \arrow[d, dashed] \arrow[ddll, bend left, "\beta"] & \Sigma I_s \\
 & \Sigma^2 Y_{s+2} \arrow[r, equal] \arrow[dl, "\Sigma^2 p_{s+2}"] & \Sigma^2 Y_{s+2} & \\
\Sigma^2 I_{s+2} & & &
\end{tikzcd}
$$

with all rows and columns triangles. Define β as shown. Ponder.

Higher Toda brackets

McKeown, nLab, 2012

Definition. Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3$, define the Toda family $\mathrm{T}(f_3, f_2, f_1)$ to consist of all pairs $(\beta, \Sigma\alpha)$, where α and β appear in a commutative diagram

$$\begin{tikzcd}
 & & \Sigma X_0 \arrow[r, "-\Sigma f_1"'] \arrow[d, "\Sigma\alpha"'] & \Sigma X_1 \arrow[d, equal] \\
X_1 \arrow[r, "f_2"] & X_2 \arrow[r] \arrow[d, equal] & C_{f_2} \arrow[r] \arrow[d, "\beta"] & \Sigma X_1 \\
 & X_2 \arrow[r, "f_3"] & X_3, &
\end{tikzcd}$$

with middle row a triangle.

Given $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n$, define the Toda bracket $\langle f_n, \ldots, f_1 \rangle \subseteq \mathcal{T}(\Sigma^{n-2} X_0, X_n)$ inductively as follows:

If $n = 2$, it is the set consisting of just the composite $f_2 f_1$.

If $n > 2$, it is the union of the sets $\langle \beta, \Sigma\alpha, \Sigma f_{n-3}, \ldots, \Sigma f_1 \rangle$, where $(\beta, \Sigma\alpha)$ is in $T(f_n, f_{n-1}, f_{n-2})$.

4-fold Toda bracket

Example. We have
$\langle f_4, f_3, f_2, f_1 \rangle = \bigcup_{\beta,\alpha} \langle \beta, \Sigma\alpha, \Sigma f_1 \rangle = \bigcup_{\beta,\alpha} \bigcup_{\beta',\alpha'} \{\beta' \circ \Sigma\alpha'\}$.

$$
\begin{tikzcd}
\Sigma^2 X_0 \arrow[r, "\Sigma\alpha'"] & C_{\Sigma\alpha} \arrow[r] \arrow[ddr, "\beta'"] & \Sigma^2 X_1 & \text{row} = -\Sigma^2 f_1 \\
\Sigma X_1 \arrow[r, "\Sigma\alpha"] & C_{f_3} \arrow[r] \arrow[u] \arrow[dr, "\beta"'] & \Sigma X_2 & \text{row} = -\Sigma f_2 \\
X_2 \arrow[r, "f_3"] & X_3 \arrow[r, "f_4"'] \arrow[u] & X_4 & \\
& 0 \arrow[u] & &
\end{tikzcd}
$$

The middle column is what is called a filtered object by Cohen, Shipley and Sagave, and so this reproduces their definition.

Self-duality for higher Toda brackets

The definition is asymmetrical. What happens in the opposite category?

More generally, we can reduce an n-fold Toda bracket to a 2-fold Toda bracket in $(n-2)!$ ways, inserting the Toda family operation in any position.

Lemma (C-Frankland)**.** The pair $(\beta, \Sigma\alpha)$ is in $T(T(f_4, f_3, f_2), \Sigma f_1)$ iff the pair $(-\beta, \Sigma\alpha)$ is in $T(f_4, T(f_3, f_2, f_1))$.

This is stronger than saying that the two ways of computing the Toda bracket $\langle f_4, f_3, f_2, f_1 \rangle$ are negatives, and the stronger statement will be important for us.

The proof is a careful application of the octahedral axiom.

Self-duality, II

For $j_1, j_2, \dots, j_{n-2}$ with $0 \le j_i < i$, write

$$T_{j_1}(T_{j_2}(T_{j_3}(\cdots T_{j_{n-2}}(f_n, \dots, f_1)\cdots)))$$

for the subset obtained by applying T in the spot with j_{n-2} maps to the left, then applying T in the spot with j_{n-1} maps to the left, etc.

Our original definition corresponds to $T_0(T_0(\cdots T_0(f_n, \dots, f_1)\cdots))$.

Theorem (C-Frankland)**.** If you compute the Toda bracket using the sequence $j_1, j_2, \dots, j_{n-2}$, it equals the original Toda bracket up to the sign $(-1)^{\sum j_i}$.

Proof. One can give an inductive argument showing that the Lemma lets you convert any such sequence into any other, using the "move" $j, j \longleftrightarrow j, j+1$. Animation: `http://turl.ca/todaanim` The move changes the sign and the parity of the sum. $\square$

Corollary. The higher Toda brackets are self-dual up to sign.

Adams d_r in terms of Toda brackets

Recall:

Proposition (C-Frankland). $d_2[x] = \langle d_1, p_{s+1}, \delta_s x\rangle = \langle d_1 \overset{\beta}{,} d_1, x\rangle$.

Theorem (C-Frankland). d_r can be expressed in terms of $(r+1)$-fold Toda brackets as:

$$d_r[x] = \langle d_1, d_1, \ldots, d_1, p_{s+1}, \delta_s x\rangle = \langle d_1, d_1, \ldots, d_1, x\rangle_{\text{fixed}}$$

The first equality is straightforward, using the dual Toda bracket.

In the second equality, "fixed" means that you choose a particular filtered object derived from the Adams resolution, which fixes all of the choices except the very last β.

(The details will be posted on the arxiv by the end of October.)

An example in the stable module category

Let $R = kC_4 = k[x]/x^4$ with char $k = 2$.
Let $M = R/x^2$. In $\mathrm{StMod}(R)$, $\Omega M = M$.
With respect to the projective class generated by k,

$$\begin{tikzcd}
M \arrow[rr, "\mu_x", "\circ" marking] & & M \arrow[dl, "\delta"] \arrow[rr, "\mu_x", "\circ" marking] & & M \arrow[dl, "\delta"] \arrow[rr, "\mu_x", "\circ" marking] & & M \arrow[dl, "\delta"] \arrow[r, "\circ" marking] & \cdots \\
& k \oplus \Omega k \arrow[ul, "p"] & & k \oplus \Omega k \arrow[ul, "p"] & & k \oplus \Omega k, \arrow[ul, "p"] & &
\end{tikzcd}$$

is an Adams resolution of M, for certain p and δ.

Given any non-zero map $\kappa : k \oplus \Omega k \to M$, one can show that $d_2[\kappa]$ has no indeterminacy, while $\langle \kappa, d_1, d_1 \rangle$ has non-trivial indeterminacy, so the containment

$$d_2[\kappa] = \langle \kappa, d_1 \overset{\beta}{,} d_1 \rangle \subseteq \langle \kappa, d_1, d_1 \rangle$$

is proper.

3-fold Toda brackets determine the triangulation

I'd like to end by advertising this nice result due to Heller (1968) with a cleaner formulation and proof due to Muro (2006 slides, 2015 e-mail):

Theorem. The diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle iff

(i) the sequence of abelian groups

$$\mathcal{T}(A, \Sigma^{-1} Z) \xrightarrow{(\Sigma^{-1} h)_*} \mathcal{T}(A, X) \xrightarrow{f_*} \mathcal{T}(A, Y) \xrightarrow{g_*} \mathcal{T}(A, Z) \xrightarrow{h_*} \mathcal{T}(A, \Sigma X)$$

is exact for every object A of $\mathcal{T}$, and

(ii) the Toda bracket $\langle h, g, f \rangle \subseteq \mathcal{T}(\Sigma X, \Sigma X)$ contains the identity map $1_{\Sigma X}$.

The proof is essentially the Yoneda Lemma and the Five Lemma.

3-fold Toda brackets determine the higher ones

Corollary. Given the suspension functor $\Sigma \colon \mathcal{T} \to \mathcal{T}$, 3-fold Toda brackets in $\mathcal{T}$ determine the triangulated structure. In particular, 3-fold Toda brackets determine the higher Toda brackets, via the triangulation.

Remark. It is unclear to us if the higher Toda brackets can be expressed directly in terms of 3-fold brackets.

Thanks for listening!