

A Characterization of Univalent Fibrations

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CT2015, Aveiro, June 18, 2015

Outline:

- Background on type theory
- Equivalence and univalence
- A characterization of univalent fibrations

Background on Type Theory

Type theory is a logical system in which the basic objects are called **types**.

Initially, types were thought of as sets, but we will think of them as being like spaces.

We write $x : X$ to indicate that x is a **term** of type X , which is analogous to the set-theoretic statement $x \in X$.

We assume given a **universe** type **Type**, and therefore can write $X : \mathbf{Type}$ to indicate that X is a type.

Each term has a **unique** type, so we can't directly talk about intersections, unions, etc. Instead, type theory comes with **type constructors** that correspond to common constructions in mathematics.

Type Constructors: Function types

For any two types A and B , there is a **function type** denoted $A \rightarrow B$, which should be thought of as an internal hom B^A .

If $f(a)$ is an expression of type B whenever a is of type A , then $\lambda a.f(a)$ denotes the function $A \rightarrow B$ sending a to $f(a)$.

Examples:

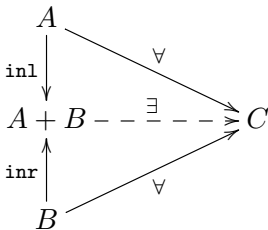
- The **identity function** id_A is defined to be $\lambda a.a$.
- The **constant function** sending everything in A to $b : B$ is $\lambda a.b$.
- Given functions $f : A \rightarrow B$ and $g : B \rightarrow C$, their **composite** gf is $\lambda a.g(f(a))$.

Type Constructors: Coproduct

Most constructions in type theory are defined [inductively](#).

For example, given types A and B , there is another type $A + B$ which is generated by terms of the form `inl a` and `inr b`.

“Generated” means that it satisfies a [weak](#) universal property:



(Using dependent types and identity types, one can [prove](#) uniqueness.)

Type Constructors: \emptyset , `unit`, \times , \mathbb{N}

There are similar definitions of:

- the **empty type** \emptyset as a weakly initial object (“free on no generators”),
- the **one point type** `unit` to be “free on one generator”,
- the **product** $A \times B$ of two types, which is generated by all pairs (a, b) , and
- the **natural numbers** \mathbb{N} , generated by $0 : \mathbb{N}$ and `succ` : $\mathbb{N} \rightarrow \mathbb{N}$.

Note the preference for constructions defined by mapping out.

Dependent Types

The above structure is enough to construct types that **depend** on terms of other types.

These **dependent types** are one of the key ideas in Martin-Löf type theory, and will play a central role in this talk.

Examples:

$\lambda a. B : A \longrightarrow \mathbf{Type}$ (a constant type family)

$\lambda n. A^n : \mathbb{N} \longrightarrow \mathbf{Type}$ (defined inductively)

$\lambda(A, B). A + B : \mathbf{Type} \times \mathbf{Type} \longrightarrow \mathbf{Type}$

Dependent Sums

Dependent sums are like the Grothendieck construction.

Given a type family $B : A \rightarrow \mathbf{Type}$, the **dependent sum** $\sum_{a:A} B(a)$ is freely generated by pairs (a, b) with $b : B(a)$.

The dependent sum has a **projection map**

$$\text{pr}_1 : \sum_{a:A} B(a) \longrightarrow A$$

sending (a, b) to a , and this can be regarded as a **fibration**.

There is also a **dependent product** $\prod_{x:A} B(x)$. A term is a function f sending each $x : A$ to an $f(x) : B(x)$.

$\prod_{a:A} B(a)$ can also be thought of as the space of **sections** of pr_1 .

Propositions as Types: Curry-Howard

A type can be thought of as a proposition, which is true when inhabited:

Types	\longleftrightarrow	Propositions
\emptyset	\longleftrightarrow	false
unit	\longleftrightarrow	true
$P \times Q$	\longleftrightarrow	P and Q
$P + Q$	\longleftrightarrow	P or Q
$P \rightarrow Q$	\longleftrightarrow	P implies Q
$\prod_{x:A} P(x)$	\longleftrightarrow	$\forall x P(x)$
$\sum_{x:A} P(x)$	\longleftrightarrow	$\exists x P(x)$

But what about the proposition $a = b$?

Identity Types

Given a type A , the **identity type** of A is a **type family** $A \times A \rightarrow \mathbf{Type}$ whose values are written $a = b$ for $a, b : A$.

This type family is generated by “reflexivity” terms of the form $\mathbf{refl}_a : a = a$ for each $a : A$.

A term of type $a = b$ was historically thought of as the assertion that a **equals** b , but in our interpretation should be thought of as a **path** from a to b in A , with \mathbf{refl}_a being the constant path at a .

The associated map $\sum_{a,b:A} (a = b) \longrightarrow A \times A$ was historically thought of as the **diagonal** map $A \rightarrow A \times A$, but for our purposes it better regarded as the **path fibration** $A^I \rightarrow A \times A$, which is obtained by replacing the diagonal map by a fibration.

Since $a = b$ is a type, *it* has an associated path type $p = q$ for $p, q : a = b$. It is **not** always the case that $p = q$.

Equivalences

We say that $f : A \rightarrow B$ is an **equivalence** if it has left and right inverses. That is,

$$\text{IsEquiv } f := \left(\sum_{g:B \rightarrow A} (gf = \text{id}_A) \right) \times \left(\sum_{h:B \rightarrow A} (fh = \text{id}_B) \right).$$

The type of **equivalences** from A to B is

$$A \simeq B := \sum_{f:A \rightarrow B} \text{IsEquiv } f.$$

Univalence Axiom

For types A and B , we define functions $\omega : (A = B) \rightarrow (A \simeq B)$ by sending refl_A to id_A .

The **Univalence Axiom** says that ω is an **equivalence** for all types A and B .

If ω is an equivalence, then there is an inverse map

$$(A \simeq B) \longrightarrow (A = B)$$

which implies that equivalent types are **equal**.

This is an assertion about the universe **Type**, and it does **not** hold in the standard set-theoretic model.

But it **does** hold for a model in simplicial sets [Kapulkin-Lumsdaine-Voevodsky] and it allows one to prove many homotopical results.

Univalent type families

We say that a type family $B : A \rightarrow \mathbf{Type}$ is **univalent** if the composite map

$$(a = a') \xrightarrow{\text{ap } B} (B(a) = B(a')) \xrightarrow{\omega} (B(a) \simeq B(a'))$$

is an **equivalence**, where $\text{ap } B$ sends refl_a to $\text{refl}_{B(a)}$.

The Univalence Axiom is the special case where the type family is $\text{id}_{\mathbf{Type}} : \mathbf{Type} \rightarrow \mathbf{Type}$.

Our goal is to **characterize** the univalent type families, as a way to better understand the Univalence Axiom.

The Case of Simplicial Sets

In the simplicial model, the notions of homotopy and equivalence match the standard notions, and a type family corresponds to a Kan fibration, obtained using the dependent sum construction.

Associated to a Kan fibration $E \rightarrow B$ is a fibration $Eq(E) \rightarrow B \times B$ whose fibre over (b, b') is the simplicial set of equivalences from the fibre E_b to the fibre $E_{b'}$.

The original fibration is univalent iff a certain natural map from the path space B^I to $Eq(E)$ is an equivalence.

Examples:

- The identity map $X \rightarrow X$ is univalent iff X is empty or contractible.
- The double-cover $S^\infty \rightarrow \mathbb{R}P^\infty$ is univalent.

$\mathbf{BAut}(F)$

We next construct a univalent type family associated to any type F .

Define

$$\mathbf{BAut}(F) := \sum_{Z:\mathbf{Type}} |F \simeq Z|,$$

where $|P|$ denotes the **propositional truncation** of the type P .

This is a type which is either empty or contractible, and has the same truth value as P .

Note that $\mathbf{BAut}(F)$ has a natural **basepoint** (F, id_F) .

Define

$$\mathbf{Aut}(F) := (F \simeq F),$$

the **monoid of self-equivalences of F** .

Lemma (Easy). $\Omega\mathbf{BAut}(F) \simeq \mathbf{Aut}(F)$, where $\Omega X := (x_0 = x_0)$.

I don't know how to define BG in general, but the above implies that this gives the correct result in a model.

The universal fibration with fibre F

Since $\mathbf{BAut}(F) := \sum_{Z:\mathbf{Type}} |F \simeq Z|$, we have a canonical map

$$\alpha := \mathbf{pr}_1 : \mathbf{BAut}(F) \longrightarrow \mathbf{Type}.$$

Proposition. The type family α is univalent for any type F .

A somewhat technical argument was sketched by [Moerdijk](#) for simplicial sets in a talk. It is proved for ∞ -topoi by [Gepner and J. Kock](#). In fact, the proposition is easy to prove in type theory (see the [HoTT library](#)), and implies the result in simplicial sets and other models.

Consider the associated fibration
$$\sum_{(Z,e):\mathbf{BAut}(F)} Z \rightarrow \mathbf{BAut}(F).$$

Taking $F = \mathbf{unit}$ gives the identity map on a contractible space, and taking $F = \mathbf{unit} + \mathbf{unit}$ gives $S^\infty \rightarrow \mathbb{R}P^\infty$.

A Characterization of Univalent Fibrations

Theorem (C). If $B : A \rightarrow \mathbf{Type}$ is a univalent type family and A is connected, then there is a type F and an equivalence $f : A \simeq \mathbf{BAut}(F)$ such that

$$\begin{array}{ccc} A & \xrightarrow{B} & \mathbf{Type} \\ f \downarrow \sim & & \nearrow \alpha \\ \mathbf{BAut}(F) & & \end{array}$$

commutes up to homotopy.

More generally, if A is not connected, then it is a coproduct of a set-indexed family of $\mathbf{BAut}(F)$'s, with pairwise non-equivalent F 's, with a similar commuting diagram.

The case of the empty coproduct gives $\emptyset \rightarrow \emptyset$, our other example.

Consequences

- Univalent fibrations are exactly the **classifying bundles for fibrations with fibre F** (or an appropriate coproduct of such). This is a notion from the 1950's that is extremely well-studied in topology.
- If the Univalence Axiom holds, then $\text{id}_{\text{Type}} : \text{Type} \rightarrow \text{Type}$ is a univalent type family, so **Type is equivalent to $\sum \text{BAut}(F)$** , where the sum is over equivalence classes of $F : \text{Type}$. We can also describe the total space of the universal fibration over **Type**.
- A given type A is **rarely** the base of a univalent fibration, and when it is, it is usually the base of only **one** univalent fibration up to equivalence. However, **coincidences** can occur. E.g. taking F to be **unit** gives $\text{unit} \rightarrow \text{unit}$, but taking F to be \emptyset gives $\emptyset \rightarrow \text{unit}$, so both are univalent.

I know of no other coincidences!

About the Proof

It's almost tautologous that a univalent family $B : A \rightarrow \mathbf{Type}$ factors through $\mathbf{BAut}(F)$ when B is connected. But there is some work in showing that the map $A \rightarrow \mathbf{BAut}(F)$ is an equivalence.

The proof has been formalized in Coq.

I also have a more complicated proof for simplicial sets, that doesn't use a universe or univalence. This could possibly give another way to prove that simplicial sets [has](#) a univalent universe, but there are non-trivial issues of strictness.

To learn more about homotopy type theory: These slides and a longer intro to type theory are on my web site.

[Mike Shulman](#)'s slides from two series of lectures are great.

Thanks for listening!