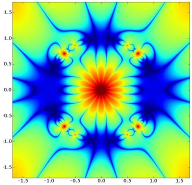
Computations involving spin networks, spin foams, quantum gravity and lattice gauge theory



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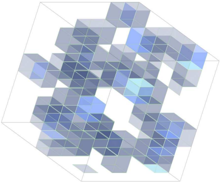


ILQGS
30 Oct 2007

Outline:

- Barrett-Crane model: behaviour, positivity, q-deformed version
- $10j$ symbol: asymptotics, graviton propagator
- Numerical comparison of new vertex proposals
- Lattice gauge theory using spin foam methods

The Riemannian Barrett-Crane model

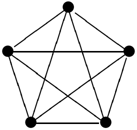
Let Δ be a triangulation of a closed 4-manifold. $\mathcal{F}$ = dual faces = triangles, $\mathcal{E}$ = dual edges = tets, $\mathcal{V}$ = dual vertices = 4-simplices.

A spin foam F is an assignment of a spin j_f to each dual face $f \in \mathcal{F}$.

The amplitude of F is

$$\mathcal{A}(F) := \left(\prod_{f\in\mathcal{F}} \mathcal{A}_f\right)\left(\prod_{e\in\mathcal{E}} \mathcal{A}_e\right)\left(\prod_{v\in\mathcal{V}} \mathcal{A}_v\right), \qquad (1)$$

where

$\mathcal{A}_v =$  $=$ 10j symbol (2)

and $\mathcal{A}_e$ and $\mathcal{A}_f$ are normalization factors that depend on the version of the Barrett-Crane model chosen.

Early computations

gr-qc/0202017, Baez-C-Halford-Tsang

Take Δ to be the simplest triangulation of the 4-sphere, as the boundary of the 5-simplex.

Using the Metropolis algorithm, we computed the expectation value of the average area of a triangle:

$$\langle O \rangle = \frac{\sum_F O(F)\mathcal{A}(F)}{\sum_F \mathcal{A}(F)} \qquad \text{where} \qquad O(F) = \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} \sqrt{j_f(j_f+1)}$$

The results showed very strong dependence on the normalization factors $\mathcal{A}_e$ and $\mathcal{A}_f$:

- For the Perez-Rovelli model, spin zero dominance.
- For the De Pietri-Freidel-Krasnov-Rovelli model, divergence.

Positivity

gr-qc/0110044, Baez-C

It was only after doing the above computations that it dawned on us that the amplitudes we were computing were always positive real numbers!

At first we suspected an error, but eventually we proved mathematically that this is correct.

From a computational point of view, this was good news, because it meant that there was no sign problem in the Metropolis algorithm.

But conceptually it raised lots of questions as it meant that there was no interference in the path integral. This highlighted the interpretation of the path integral as a projection onto physical states.

q-deformed version

arXiv:0704.0278, CQG, C-Khavkine

The q-deformed Barrett-Crane model replaces the group SU(2) by the quantum group $SU_q(2)$. When $q = \exp(i\pi/r)$ is a root of unity, this regularizes the theory by eliminating spins greater than $(r-2)/2$. As $r \to \infty$, $q \to 1$, the undeformed value.

Also, it has been suggested by Smolin that r is related to the cosmological constant:

$$\Lambda \sim 1/r.$$

We have recently done computations of expectation values which greatly generalize earlier work:

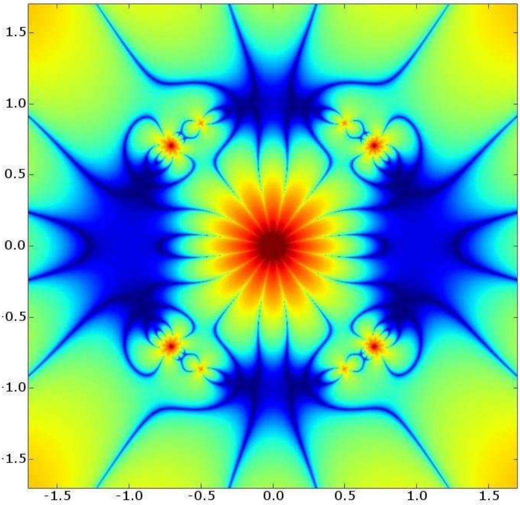
- The deformation parameter q can be varied.
- The triangulation can be varied, and can be large.
- Several different observables have been used.

q-deformed spin network evaluations

arXiv:0704.0278, C-Khavkine

The first step was generalizing the spinnet library to handle $SU_q(2)$. It can now handle:

- $q = 1$ (classical case)
- $q = m/n$ an exact rational number
- q a floating point real number
- $q = \exp(i\pi/r)$ a root of unity
- q a floating point complex number
- symbolic q



The plot shows the real part of the tet network, with all spins equal to 2.

Observables

arXiv:0704.0278, C-Khavkine

Spin foam observables depend on face spin labels:

average spin
$$J(F) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lfloor j_f \rceil \,, \tag{3}$$

spin variance
$$(\delta J)^2(F) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \left(\lfloor j_f \rceil - \langle J \rangle \right)^2 \,, \tag{4}$$

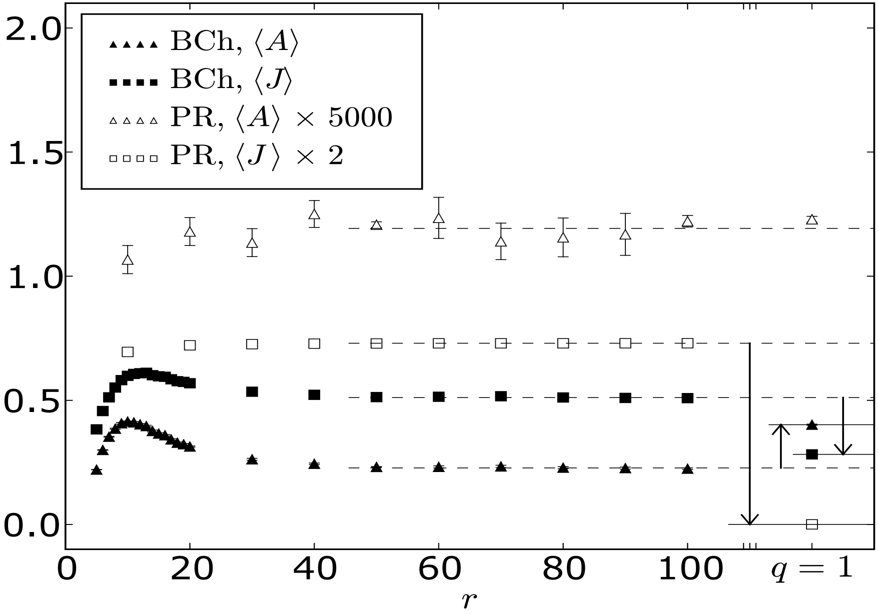
average area
$$A(F) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sqrt{\lfloor j_f \rceil \, \lfloor j_f + 1 \rceil}, \tag{5}$$

spin-spin corr.
$$C_d(F) = \frac{1}{N_d} \sum_{\mathrm{dist}(f,f')=d} \frac{\lfloor j_f \rceil \, \lfloor j_{f'} \rceil - \langle J \rangle^2}{\langle (\delta J)^2 \rangle}. \tag{6}$$

Quantum half integers $\lfloor j \rceil = j$ when $q = 1$, but $\lfloor j \rceil \sim \sin(2j\pi/r)$ when $q = e^{i\pi/r}$.

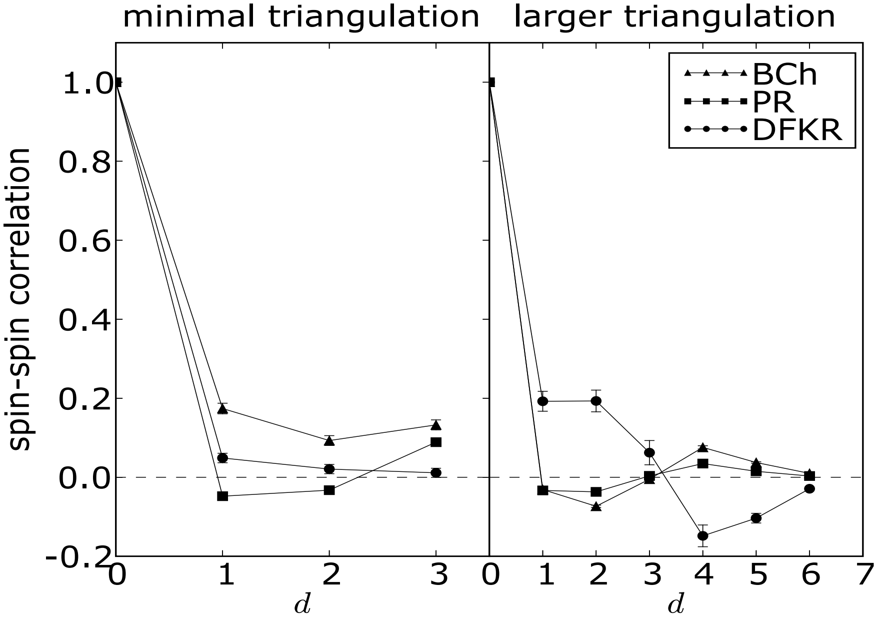
Expectation values

arXiv:0704.0278, CQG, C-Khavkine



Spin Correlation

arXiv:0704.0278, C-Khavkine



Asymptotics

gr-qc/9809032, Barrett-Williams

Since the $10j$ symbol is the key ingredient of the Barrett-Crane model, it has been well studied. It can be computed as an integral:

$$\{10j\} = \pm \int_{S^3} \int_{S^3} \int_{S^3} \int_{S^3} \int_{S^3} \prod_{1 \le k < l \le 5} K_{j_{kl}}(\phi_{kl})\, dx_1 \cdots dx_5, \qquad (7)$$

where ϕ_{kl} is the angle between the unit vectors x_k and x_l, and

$$K_j(\phi) := \frac{\sin((2j+1)\phi)}{\sin(\phi)}. \qquad (8)$$

The spins j_{kl} label the triangles of a 4-simplex, giving them each area $2j_{kl} + 1$. The x_k can be thought of as normals to the 5 tetrahedra.

Barrett and Williams studied this integral for large spins. They showed that the stationary phase points correspond to 4-simplices with the prescribed triangle areas (up to scale) and that these points contribute according to the Regge action.

Degenerate points gr-qc/0208010, Baez-C-Egan; Barrett-Steele; Freidel-Louapre

As the spins are scaled by a factor λ, the contribution from the stationary phase points goes like $\lambda^{-9/2}$.

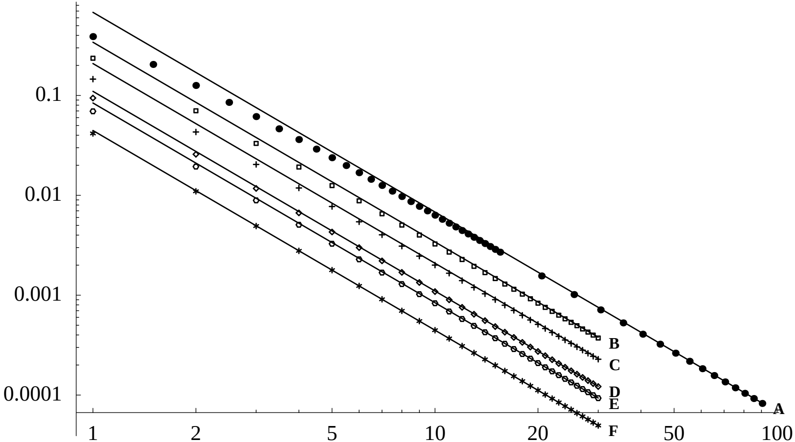
We performed computations to verify that the $10j$ symbol behaved asymptotically like the Regge action, and found that this was false. We observed that the $10j$ symbol goes like λ^{-2}, with no oscillation.

Further analytic study (by several independent groups) showed that this is due to contributions from degenerate 4-simplices, i.e. flat 4-simplices with zero volume. These were noticed but not studied by Barrett and Williams.

This has lead to new proposals for the vertex amplitude in quantum gravity.

Asymptotics

gr-qc/0208010, Baez-C-Egan



The points show the numerical evaluation of six different $10j$ symbols as the scale factor λ (x-axis) is varied. The lines show the asymptotic predictions using degenerate points.

Graviton Propagator Rovelli, Bianchi, Modesto, Speziale, Livine, Willis, C, ...

Rovelli and others proposed a way to define 2-point functions in the Barrett-Crane model. The leading contribution is of the form

$$W_{ab} = \frac{\displaystyle\sum_{\{j_k\}} h(j_a)\, h(j_b)\, \Psi[j]\, \{10j\}}{\displaystyle\sum_{\{j_k\}} \Psi[j]\, \{10j\}}, \qquad h(j) = j(j+1) - j_0(j_0+1)$$

The sum is over ten spins labelling the triangles of a 4-simplex.
$h(j_a)h(j_b)$ is the field insertion. Ψ is a chosen boundary state.
$\{10j\}$ denotes the $10j$ symbol.

Graviton Propagator

Rovelli, Bianchi, Modesto, Speziale, Livine, Willis, C, ...

More concisely:

$$W_{ab} = \frac{1}{\mathcal{N}} \sum_{\{j_k\}} h(j_a)\, h(j_b)\, \Psi[j]\, \{10j\}, \qquad h(j) = j(j+1) - j_0(j_0+1)$$

Rovelli and Speziale proposed a Gaussian boundary state:

$$\Psi[j] = \exp\left(-\frac{1}{2j_0} \sum_{i,k} \alpha_{ik}(j_i - j_0)(j_k - j_0) + i\Phi \sum_k j_k \right) \qquad (9)$$

peaked around a regular 4-simplex, where α_{ik} is a 10x10 matrix of real numbers. Here j_0 determines the areas of the triangles of the regular 4-simplex, and $\Phi = \arccos(-1/4)$ is the dihedral angle.

For large j_0, W_{ab} is expected to go as $1/j_0$, and Rovelli argued that this is indeed the case.

In numerical computations it was difficult to see this behaviour because the computations were too difficult.

Computations for diagonal α_{ik} arXiv:0710.0617, C-Livine-Speziale

If we restrict to $\alpha_{ik} = \alpha\delta_{ik}$, a diagonal matrix, then the boundary state factors:

$$\Psi[j] = \exp\left(-\frac{\alpha}{2j_0}\sum_k (j_k - j_0)^2 + i\Phi\sum_k j_k \right) = \prod_k \exp\left(-\frac{\alpha}{2j_0}(j_k - j_0)^2 + i\Phi j_k \right) \tag{10}$$

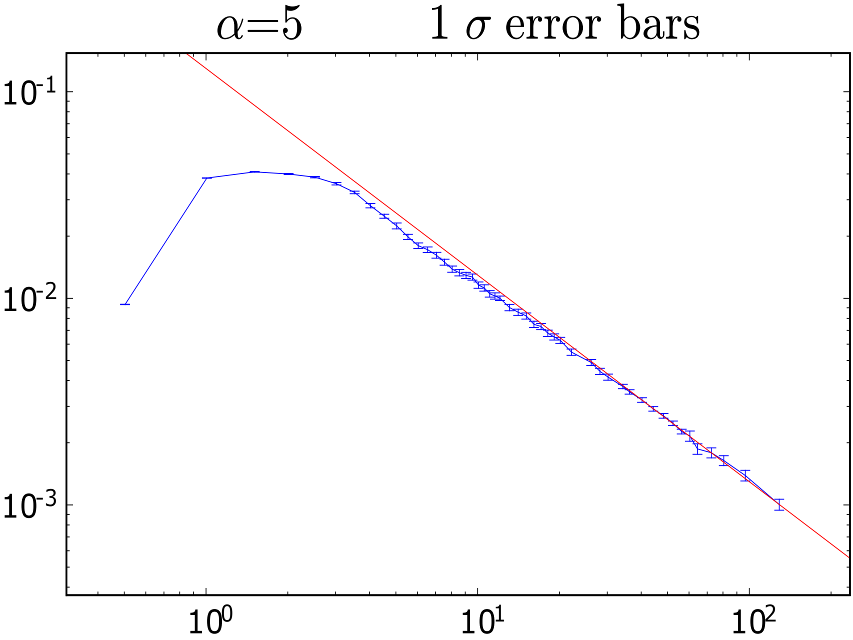
As mentioned earlier, the $10j$ symbol can be expressed as an integral whose integrand is a product of kernels, one for each k.

Thus if we exchange the order of summation and integration, the ten nested summations become a product of ten independent summations.

This allows us to compute the propagator for diagonal α:

Computations for diagonal α_{ik}

arXiv:0710.0617, C-Livine-Speziale



New vertex proposals

Engle-Pereira-Rovelli, Livine-Speziale, Freidel-Krasnov

Motivated by problems found with the Barrett-Crane vertex, new vertex amplitudes have been proposed:

- Engle, Pereira & Rovelli
  (2007, arXiv:0705.2388, arXiv:0708.1236)
- Livine & Speziale
  (2007, arXiv:0708.1915)
- Freidel & Krasnov
  (2007, arXiv:0708.1595)

Briefly, the idea is to impose constraints weakly in the quantum theory.

Comparison of labellings

Khavkine

Barrett-Crane:

Spins j_f labelling the triangles. $10j$ per vertex.

Engle-Pereira-Rovelli, Livine-Speziale:

In addition, intertwiners i_e labelling the tetrahedra. $10j + 5i$ per vertex.

Freidel-Krasnov:

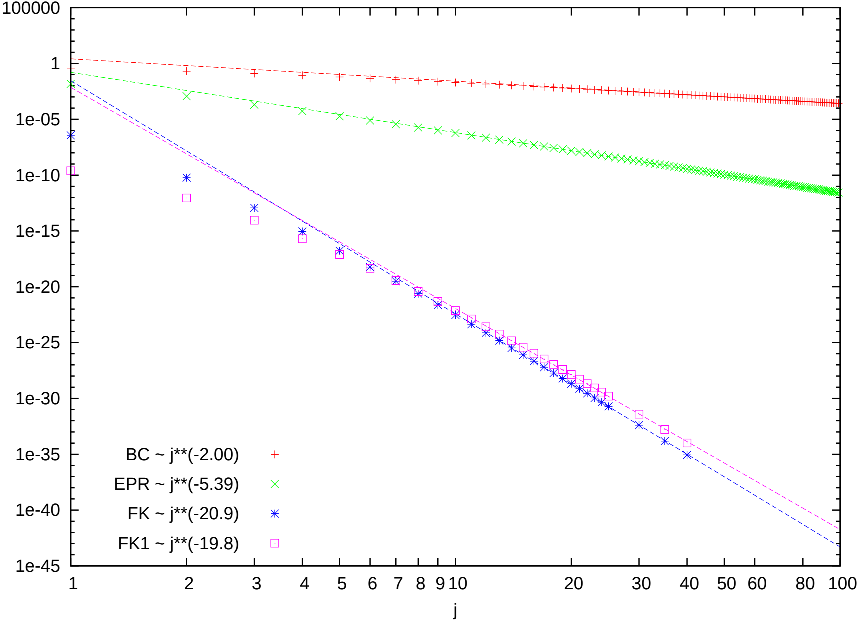
In addition, intertwiners $k_{e,f}$ for each choice of triangle and tetrahedron containing that triangle. $10j + 5i + 20k$ per vertex.

Goal: compare these models by computing expectation values of observables.

First step: compare the vertex amplitudes.

Results

Khavkine



Lattice Gauge Theory

Conrady, Oeckl, Pfeiffer, . . .

Many people have observed that spin foam methods can be used to provide a dual formulation of pure Yang-Mills lattice gauge theory.

This is an exact duality. It replaces integrations over group variables labelling edges with summations over representation variables labelling edges and plaquettes (faces).

The terms in the summation involve evaluating complicated spin networks.

Why dualize?

- Gives a gauge-invariant picture.
- May be computationally faster in some contexts.
- Bridge to quantum gravity with matter.
- May help with hard problems in LGT, e.g. dynamical fermions.

Duality transformation

Conventional lattice gauge theory:

$$\mathcal{Z} = \int \prod_{p\in\mathcal{P}} e^{-S(g_p)} \prod_{e\in\mathcal{E}} dg_e \tag{11}$$

where g_p is the product of the group elements labelling the edges of the plaquette p. If you expand the action in terms of characters

$$e^{-S(g)} = \sum_j c_j \chi_j(g) \tag{12}$$

and exchange the order of integration and summation, then

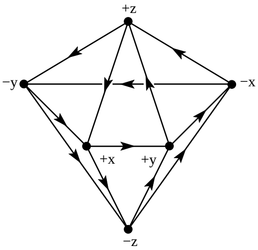
$$\mathcal{Z} = \int \prod_{p\in\mathcal{P}} \sum_{j_p} c_{j_p}\chi_{j_p}(g_p) \prod_{e\in\mathcal{E}} dg_e = \sum_{\{j_p\}} \int \prod_{p\in\mathcal{P}} c_{j_p}\chi_{j_p}(g_p) \prod_{e\in\mathcal{E}} dg_e$$

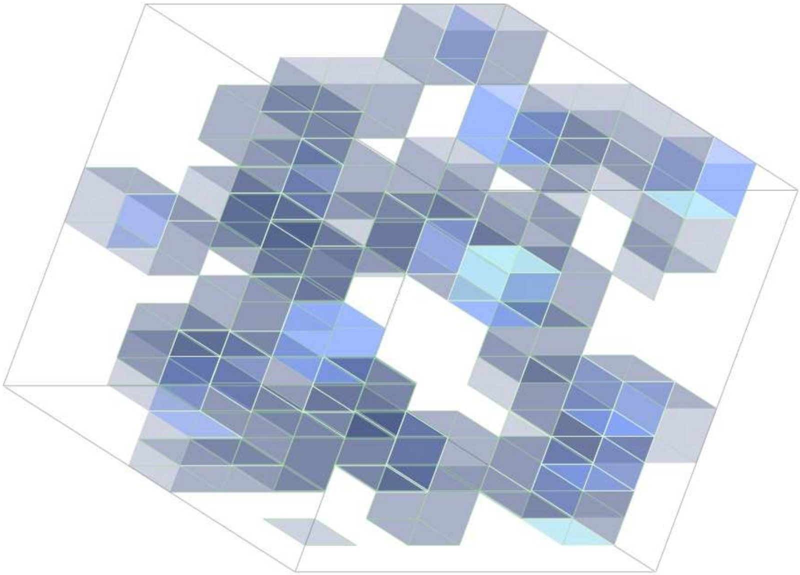
Spin foam formulation

Specializing to the case of $D = 3$ and $G = SU(2)$, we can factor the characters into contributions from each g_e and find

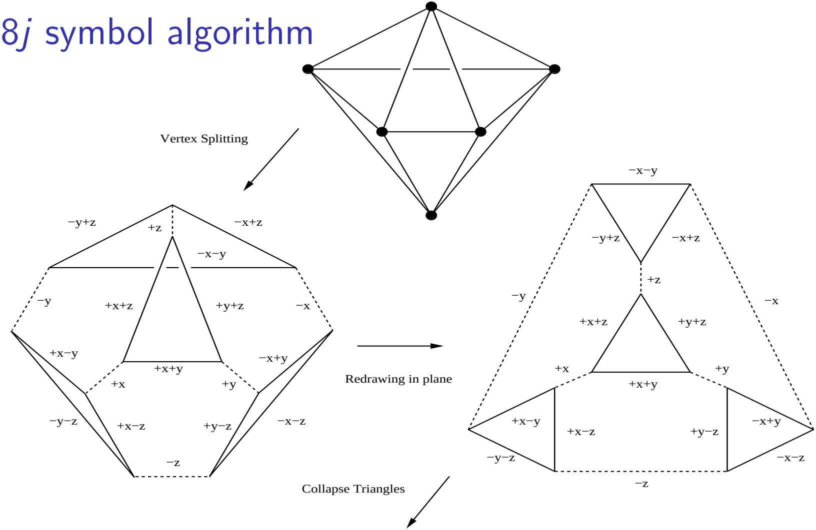
$$\mathcal{Z} = \sum_{\{j_p\}} \sum_{\{i_e\}} \prod_{v \in \mathcal{V}} 18j^v(i_v, j_v) \prod_{e \in \mathcal{E}} N^e(i_e, j_e)^{-1} \left(\prod_{p \in \mathcal{P}} e^{-\frac{2}{\beta} j_p(j_p+1)} (2j_p + 1) \right) .$$

Here $18j^v(i_v, j_v)$ is the 18j symbol:



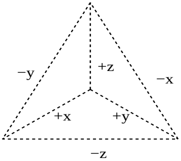


Fast 18j symbol algorithm



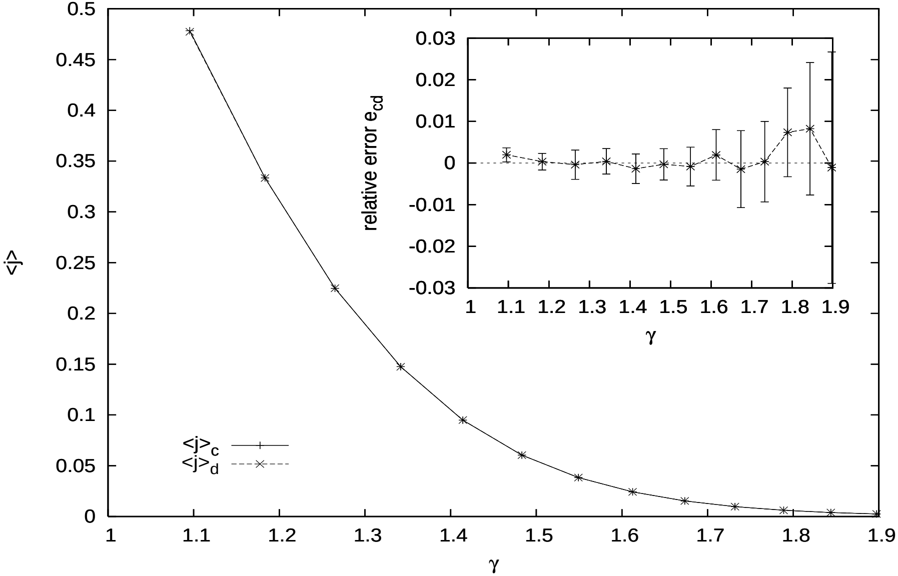
$$\frac{\begin{bmatrix} i_{+x} & j_{+x+z} & i_{+z} \\ j_{+y+z} & i_{+y} & j_{+x+y}\end{bmatrix}}{\theta(i_{+x},i_{+y},i_{+z})} \frac{\begin{bmatrix} i_{+z} & j_{-y+z} & i_{-y} \\ j_{-x-y} & i_{-x} & j_{-x+z}\end{bmatrix}}{\theta(i_{+z},i_{-x},i_{-y})}$$

$$\times \frac{\begin{bmatrix} i_{+x} & j_{+x-z} & i_{-z} \\ j_{-y-z} & i_{-y} & j_{+x-y}\end{bmatrix}}{\theta(i_{+x},i_{-y},i_{-z})} \frac{\begin{bmatrix} i_{+y} & j_{-x+y} & i_{-x} \\ j_{-x-z} & i_{-z} & j_{+y-z}\end{bmatrix}}{\theta(i_{+y},i_{-z},i_{-x})}$$



Computations

arXiv:0705.2629 **v2**, Phys. Rev. D, Cherrington-C-Khavkine



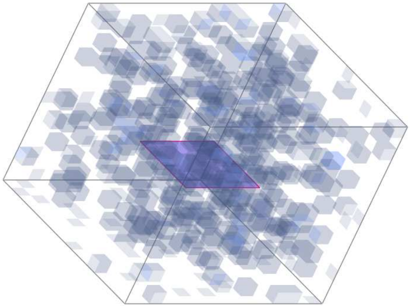
Pure SU(2) Yang-Mills on 8^3 lattice

Current and future LGT work

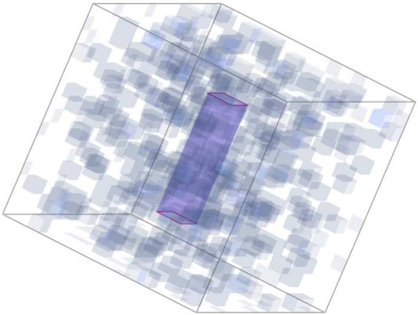
- $D = 4$
- Higher gauge groups, such as $SU(3)$
- Wilson loop observables
- Dynamical fermions
- Gauge theory coupled to quantum gravity (Oriti, Pfeiffer, Speziale, . . .)

Wilson loop observables

Cherrington, in prep



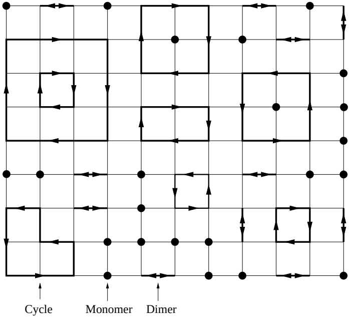
Confinement



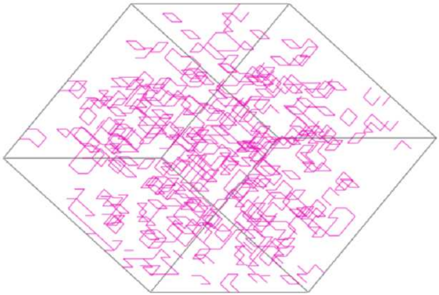
2 point correlation function

Dynamical fermions

arXiv:0710.0323, Nucl. Phys. B, Cherrington



D=2, G=U(1)



D=3, G=SU(2)

Conclusions

- Computation has repeatedly lead to new and often unexpected insights.
- These facts are often then derived analytically.
- The results of computation can help choose between existing models and can suggest new models.
- Computational techniques from one area (e.g. spin foams and spin networks) can be effective in another area (e.g. lattice gauge theory).