

Representation theory, Homework Set 2

Here is the complete list of questions for HW2, which is due Tuesday, October 6 at 2pm.

Please put each solution on its own page (but different parts (a), (b), etc. can be on the same page).

Remember that homework is graded on both **clarity** and correctness, and that clarity means giving an appropriate amount of detail, not too much and not too little. Good notation and a good choice of proof strategy help as well.

Use the textbook proofs as a good guide to style.

Q1: Consider the representation of $G = \mathbb{Z}/4$ on $V = \mathbb{R}^2$, where the generator acts via rotation by 90 degrees counterclockwise. This is a *real* representation. (You don't need to prove this.)

- (a) (1 mark) Show that it is irreducible.
- (b) (2 marks) Show that $\dim_{\mathbb{R}} \operatorname{Hom}_G(V, V) > 1$.

Q2: Consider the equilateral triangle in $\mathbb{R}^2$ with vertices

$$v_1 = (1, 0), \quad v_2 = (-1/2, \sqrt{3}/2) \quad \text{and} \quad v_3 = (-1/2, -\sqrt{3}/2).$$

The group S_3 permutes these vertices by $\sigma(v_i) = v_{\sigma(i)}$ and this extends linearly to give an action φ on $\mathbb{R}^2$. Consider the matrix form of this action with respect to the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and consider these matrices as being in $GL_2(\mathbb{C})$, so $\varphi: S_3 \rightarrow GL_2(\mathbb{C})$. Since the action preserves lengths, the matrices are unitary. For example, since $\varphi((23))$ is reflection in the x -axis,

$$\varphi((23)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- (a) (2 marks) Compute all six matrices of φ .
- (b) (2 marks) Verify for yourself that the four functions $\varphi_{ij}: G \rightarrow \mathbb{C}$ you obtain are orthogonal and have the correct norm. Only show me the computations for $\langle \varphi_{11}, \varphi_{11} \rangle$ and $\langle \varphi_{11}, \varphi_{12} \rangle$.
- (c) (2 marks) Use the character χ_φ to show that φ is irreducible and is isomorphic to the 2-dimensional representation ρ from Example 3.1.14.

Q3: Let $\alpha: S_n \rightarrow GL_n(\mathbb{C})$ be the standard representation from Example 3.1.9.

- (a) (1 mark) Show that $\chi_\alpha(\sigma)$ equals the number of i such that $\sigma(i) = i$.
- (b) (2 marks) (The remaining parts take $n = 4$.) List one representative of each conjugacy class of S_4 and compute the number of elements in each conjugacy class.
- (c) (2 marks) Compute χ_α on each representative from (b) and determine how many copies of the trivial representation occur in α .
- (d) (2 marks) Use $\langle \chi_\alpha, \chi_\alpha \rangle$ to determine how many other irreps occur in α , and deduce what their characters must be.