Representation theory, Homework Set 3

Due Thursday, October 22 at 2pm.

We have already covered all of the relevant material.

Q1 (1+2+1+2 marks): Exercise 4.11. (If needed for your solution, be sure to show that the representations described in the question are not isomorphic. Note that $M_{mn}(\mathbb{C})$ means $m \times n$ matrices.)

For the next question, we define the quaterion group to be the group generated by symbols $\hat{i}, \hat{j}, \hat{k}$ with the relations $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k}$. The symbol -1 is then *defined* to be the common value, and it *follows* that (-1) is central and $(-1)^2 = 1$. The notation $-\hat{i}$ is then short for $(-1)\hat{i}$, and similarly for the other generators. This group has eight elements. You can make use of these facts without proving them.

Q2 (2+2+1+1 marks): Exercise 4.12. (For part 1, be sure to show that ρ is a representation. For part 3, you can save work, since you can show from the previous parts that there are five inequivalent irreps.)

Q3 (4 marks): Exercise 5.13.1. (Part 1 only. Show it is well-defined and an isomorphism of groups.)

The rest are not to be handed in, but are for practice/interest.

N1: Exercise 5.6.

N2: Let G be a finite group. Consider the action of $G \times G$ on G by $(g, h)k = gkh^{-1}$. This gives a representation of $G \times G$ on the group algebra $\mathbb{C}G$. Since $\mathbb{C}G$ is isomorphic to $L(G)$, $L(G)$ also has an action of $G \times G$ on it, making it into a $(G \times G)$-representation.

(a) Describe explicitly how $G \times G$ acts on $L(G)$.

Now suppose that $\varphi : G \to GL_d(\mathbb{C})$ is a representation of G. Then $M_d(\mathbb{C})$ is a representation of $G \times G$ via

$$(g, h)A = \varphi_g A \varphi_h^{-1}$$

[This is similar to 4.11, but slightly different.]

(b) Show that the map $T : L(G) \to M_d(\mathbb{C})$ sending f to

$$\sum_{g \in G} f(g)\, \varphi_g$$

is a $(G \times G)$-morphism.

Notes about N2:

If φ is an irreducible G rep, then the above makes $M_d(\mathbb{C})$ into an irreducible $G \times G$ rep (proof similar to 4.11), and so one way to express Wedderburn's Thm (5.5.6) is that we have decomposed $L(G)$ into irreps with respect to the $G \times G$ action.

Moreover, if you only look at how the left factor of $G \times G$ acts, you see that $M_d(\mathbb{C})$ is isomorphic as a G-rep to d copies of the representation φ. Combining this with the isomorphism of $\mathbb{C}G$ with $L(G)$, one recovers the decomposition of the regular representation into d_k copies of each irrep $\varphi^{(k)}$. So Wedderburn's Theorem is a stronger version of this decomposition, which also takes into account the right action of G on itself.