## §6.2 d|n

**Lemma:** $A \in GL_d(\mathbb{C})$, $A^n = I$. Then $A$ is diagonalizable and the e'values are $n^{th}$ roots of unity.

(Book: 4.1.10 using rep. $\mathbb{C}_\chi$.)

**Pf:** Minimal poly of $A$ divides $z^n - 1$. But $z^n - 1$ has no repeated roots, so same is true of min. poly. $\therefore$ d'ble. E'values are roots of min. poly, so they are roots of $z^n - 1$. □

**Cor 6.2.1:** $\mathcal{Q}$ a rep of a finite group $G$. Then $\chi_\mathcal{Q}(g) \in \mathbb{A}$ $\forall g \in G$.

**Pf:** Let $n = |G|$. Then $(\mathcal{Q}_g)^n = I$. By Lemma, e'values of $\mathcal{Q}_g$ are roots of unity, so in $\mathbb{A}$. So

$$\chi_\mathcal{Q}(g) = \mathrm{Tr}(\mathcal{Q}_g) = \text{sum of e'values (with mult)} \in \mathbb{A}$$

by 6.1.6. □

**Thm 6.2.3:** $\mathcal{Q}$ an irrep of $G$ of degree $d$, $g \in G$, $h$ = size of conj. class of $g$. Then

$$\frac{h\chi_\mathcal{Q}(g)}{d} \in \mathbb{A}$$

**Pf:** Next class.

**Thm 6.2.4:** (Dimension Thm, Frobenius) $\mathcal{Q}$ irrep of degree $d$, then $d|n$. $n = |G|$.

**Pf:** Let conj. classes of $G$ be $C_1, \dots, C_s$ with sizes $h_1, \dots, h_s$. Let $\chi_i = \chi_\mathcal{Q}(g)$ for $g \in C_i$.

$$1 = \langle \chi_\mathcal{Q}, \chi_\mathcal{Q} \rangle = \frac{1}{n} \sum_{k=1}^{s} h_k \chi_k \bar{\chi}_k$$

So

$$\frac{n}{d} = \sum_{k=1}^{s} \underbrace{\left(\frac{h_k \chi_k}{d}\right)}_{\in \mathbb{A} \text{ by 6.2.3.}} \bar{\chi}_k \in \mathbb{A} \text{ by 6.2.1.}$$

So $\frac{n}{d} \in \mathbb{A}$. So $\frac{n}{d} \in \mathbb{Z}$. So $d|n$. □