Ext groups in Homotopy Type Theory

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Outline:

- Introduction and motivation
- Ext in HoTT
- Ext in an ∞-topos

Based on arxiv:2305.09639. These slides are on my web site.

Introduction and motivation

Homological algebra is a fundamental tool in mathematics, with applications in algebra, homotopy theory and algebraic geometry.

Ext groups of R-modules are examples of derived functors, from which other derived functors can be computed.

They are also important algebraic invariants, and appear in the classical universal coefficient theorem for cohomology.

We have developed the basic theory of Ext groups in HoTT, and have investigated the interpretation of these results into an ∞-topos.

This talk will focus on the aspects that are unique in this setting.

Ext^1 in HoTT

We work in Book HoTT, with HITs and enough univalent universes.

We will follow the resolution-free approach to Ext from [Yoneda54].

Fix a (0-truncated) ring R, and let A and B be R-modules.

Definition. Let $\mathsf{SES}_R(B, A)$ be the type of short exact sequences

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0.$$

That is,

$$\mathsf{SES}_R(B, A) :\equiv \sum_{E:\mathsf{Mod}_R} \; \sum_{i:\mathsf{Mono}_R(A,E)} \; \sum_{p:\mathsf{Epi}_R(E,B)} \mathsf{IsExact}(i, p).$$

Define

$$\mathsf{Ext}^1_R(B, A) :\equiv \|\mathsf{SES}_R(B, A)\|_0.$$

This is an abelian group under the Baer sum of extensions.

1 focused goal (shelved: 9)

Smallness

We defined $\mathsf{Ext}^1_R(B, A) :\equiv \|\mathsf{SES}_R(B, A)\|_0$, where SES quantifies over the large type of all R-modules. Thus it is a large type.

However, we prove:

Theorem. Let B and A be abelian groups. Then

$$\mathsf{SES}_{\mathbb{Z}}(B, A) \simeq \big(\mathsf{K}(B, 2) \to_* \mathsf{K}(A, 3)\big).$$

Corollary. For any ring R and R-modules A and B, $\mathsf{Ext}^1_R(B, A)$ is equivalent to a small type.

(Because the type of R-module structures on an abelian group is small.)

Projectivity

Proposition. $A \to E \xrightarrow{p} B$ is trivial in $\mathsf{Ext}^1(B, A)$ iff p merely splits.

Definition. An R-module P is projective if for every R-module E, every epimorphism $p : \mathsf{Mod}_R(E, P)$ merely splits.

Proposition. Let P be an R-module. The following are equivalent:

(i) P is projective.

(ii) For every E and B and every epimorphism $p : \mathsf{Mod}_R(E, B)$, the post-composition map $p_* : \mathsf{Mod}_R(P, E) \to \mathsf{Mod}_R(P, B)$ is an epimorphism.

(iii) $\mathsf{Ext}^1_R(P, A) = 0$ for all R-modules A.

Injective is defined dually, and the duals of both propositions hold.

Ext^n in HoTT

The definition of the higher Ext groups is more subtle. For $n \geq 1$, define:

$$\mathsf{ES}^1_R(B, A) :\equiv \mathsf{SES}_R(B, A)$$

$$\mathsf{ES}^{n+1}_R(B, A) :\equiv \sum_{C:\mathsf{Mod}_R} \mathsf{ES}^n_R(C, A) \times \mathsf{SES}_R(B, C).$$

Then we define $\mathsf{Ext}^n_R(B, A)$ to be a certain set-quotient of $\mathsf{ES}^n_R(B, A)$.

In [Flaten23], the usual long exact sequences are proved. We deduce:

Theorem. If B has a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0,$$

then $\mathsf{Ext}^n_R(B, A)$ is isomorphic to the nth cohomology of

$$\cdots \longleftarrow \mathsf{Mod}_R(P_2, A) \longleftarrow \mathsf{Mod}_R(P_1, A) \longleftarrow \mathsf{Mod}_R(P_0, A) \longleftarrow 0.$$

Ext vs cohomology

Theorem. Let X be a pointed, connected type, and let $M : X \to \mathsf{Ab}$ be a family of abelian groups. Then

$$H^1(X; M) \simeq \mathsf{Ext}^1_{\mathbb{Z}\pi_1(X)}(\mathbb{Z}, M_*).$$

Here $H^1(X; M) :\equiv \|\prod_x \mathsf{K}(M_x, 1)\|_0$.

Classically, this is true with 1 replaced by n, but we don't know how to prove this in HoTT.

It's true iff $\{H^n(X; -)\}$ forms a universal delta functor (cf. Cherubini's talk).

Ext in an ∞-topos

Fix a (Grothendieck) ∞-topos $\mathcal{X}$ with a (0-truncated) ring object R.

We write $\mathtt{Ext}^n_R(B, A) \in \mathrm{Ab}_{\mathcal{X}}$ for the interpretation of our Ext group, and $\mathrm{Ext}^n_R(B, A) \in \mathrm{Ab}$ for the usual external Ext group.

Things that can happen:

- $\mathtt{Ext}^1_{\mathbb{Z}}(B, A) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(B, A) \neq 0$. It follows that the latter isn't the global points of the former. This happens in the ∞-topos of G-spaces for any non-trivial G.
- $\mathtt{Ext}^1_{\mathbb{Z}}(B, A) \neq 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(B, A) = 0$. This happens in the Sierpiński ∞-topos.
- $\mathtt{Ext}^2_{\mathbb{Z}}(B, A) \neq 0$. Also in the Sierpiński ∞-topos.
- There can fail to be enough HoTT-projective abelian groups. This happens in the "Trimble ∞-topos".

In spaces, $\mathtt{Ext}_R$ and Ext_R agree.

Sheaf Ext

There is also a third kind of Ext group.

Definition. Define sheaf Ext $\underline{\mathrm{Ext}}^n_R(B,-) : \mathrm{Mod}_R \to \mathrm{Ab}_{\mathcal{X}}$ to be the n^{th} right derived functor of $\underline{\mathrm{Mod}}_R(B,-)$ with respect to the externally injective modules.

Definition. We say that sets cover in $\mathcal{X}$ if for every object $X \in \mathcal{X}$ there exists a 0-truncated object V and an effective epimorphism $V \to X$.

Theorem. Suppose that sets cover in $\mathcal{X}$. Then

$$\mathtt{Ext}^n_R(B,A) \simeq \underline{\mathrm{Ext}}^n_R(B,A)$$

for every R, B, A and n. If, in addition, the global points functor Γ is exact, then

$$\Gamma\,\mathtt{Ext}^n_R(B,A) \simeq \mathrm{Ext}^n_R(B,A).$$

Proof sketch

Theorem. Sets cover $\implies \mathtt{Ext}^n_R(B, A) \simeq \underline{\mathrm{Ext}}^n_R(B, A)$.

The proof relies on comparing various notions of injectivity of an R-module I:

- External injectivity: for every monomorphism $m : A \to B$, the map $m^* : \mathrm{Mod}_R(B, I) \to \mathrm{Mod}_R(A, I)$ in Ab is epi.
- Internal injectivity: for every monomorphism $m : A \to B$, the map $m^* : \underline{\mathrm{Mod}}_R(B, I) \to \underline{\mathrm{Mod}}_R(A, I)$ in $\mathrm{Ab}_{\mathcal{X}}$ is epi.
- HoTT-injectivity: the interpretation of the notion from HoTT.

Facts:

- External injectivity $\implies$ internal injectivity.
- I is HoTT-injective $\iff$ $X \times I$ is internally injective in $\mathcal{X}/X$ for all X.
- When sets cover, HoTT-injective $\iff$ internally injective.

It follows that when sets cover, we can use an externally injective resolution to compute $\mathtt{Ext}$, which therefore agrees with $\underline{\mathrm{Ext}}$.

Open questions

In HoTT:

0. Is the abelian group $\mathsf{Ext}^n_R(B, A)$ equivalent to a small type for $n \geq 2$? Is it independent of the universe for $n \geq 2$?

S0. Are injectivity and projectivity independent of the universe?

SS0. Does $H^n(X; M) \simeq \mathsf{Ext}^n_{\mathbb{Z}\pi_1(X)}(\mathbb{Z}, M_*)$ for $n > 1$?

In an ∞-topos:

SSS0. Do HoTT-injectivity and HoTT-projectivity only depend on the 1-topos of 0-truncated objects? Do they always agree with internal injectivity and internal projectivity?

SSSS0. Does the interpretation of $\mathsf{Ext}^n_R(B, A)$ depend only on the 1-topos of 0-truncated objects?

References

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Formalization: https://github.com/HoTT/Coq-HoTT and https://github.com/jarlg/Yoneda-Ext

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And more—see the paper!

Thanks!