# THE HUREWICZ THEOREM IN HOMOTOPY TYPE THEORY

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Abstract. We prove the Hurewicz theorem in homotopy type theory, i.e., that for $X$ a pointed, $(n-1)$-connected type $(n \geq 1)$ and $A$ an abelian group, there is a natural isomorphism $\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X; A)$ relating the abelianization of the homotopy groups with the homology. We also compute the connectivity of a smash product of types and express the lowest non-trivial homotopy group as a tensor product. Along the way, we study magmas, loop spaces, connected covers and prespectra, and we use 1-coherent categories to express naturality and for the Yoneda lemma.

As homotopy type theory has models in all $\infty$-toposes, our results can be viewed as extending known results about spaces to all other $\infty$-toposes.

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## 1. Introduction

Homotopy type theory is a formal system which has models in all $\infty$-toposes ([KL18], [LS19], [Shu19], [Boe20; BB])[1]. As such, it provides a convenient way to prove theorems

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[1]The initiality and semantics of higher inductive types still need to be fully worked out.

for all $\infty$-toposes. In addition, homotopy type theory is well-suited to being formalized in a proof assistant ([Bau+17], [Doo18]).

Working in homotopy type theory as described in the book [Uni13], we prove the Hurewicz theorem:

**Theorem H** (Theorem 3.12)**.** *For $n \geq 1$, $X$ a pointed, $(n-1)$-connected type, and $A$ an abelian group, there is a natural isomorphism*

$$\pi_n(X)^{ab} \otimes A \cong \tilde{H}_n(X; A),$$

*where on the left-hand-side we take the abelianization (which only matters when $n = 1$). In particular, when $A$ is the integers, this specializes to an isomorphism $\pi_n(X)^{ab} \cong \tilde{H}_n(X)$.*

As mentioned above, this holds in any $\infty$-topos, and so is more general than the well-known Hurewicz theorem in topology. Interpreting the statement in an $\infty$-topos is somewhat subtle. The groups that appear in the statement are internal group objects whose underlying object is 0-truncated (a "set", internally). The quantification over $n$ means that there is a map $h : H \to \mathbb{N}$ in the $\infty$-topos representing a family of objects over the natural numbers object, and that this map has a section. In particular, since each ordinary natural number gives a global element of $\mathbb{N}$, it follows that the fibre of $h$ over that element must itself have a global element. Continuing in this way, we deduce that for given objects $X$ and $A$ as in the statement, the two internal group objects shown are equivalent as group objects. For more on the interpretation of type theory, see [Shu15, Section 4.2] and [Shu19] for the interpretation in arbitrary $\infty$-topoi, and [KL12] for a more explicit interpretation in simplicial sets.

Since we prove this theorem for an arbitrary $\infty$-topos, we must be careful to use arguments that apply in this generality. For example, it is not true in every $\infty$-topos that a surjective map of sets has a section, so we cannot use the axiom of choice. Similarly, the law of excluded middle and Whitehead's theorem can both fail, so we must not use these results either. Because of this, our proof is different from other known proofs.

Before giving more details, we give some motivation for the interest in this result, for those less familiar with traditional homotopy theory.

**Motivation.** In topology, homotopy groups are in a certain sense the strongest invariants of a topological space, and so their computation is an important tool when trying to classify spaces up to homotopy. In homotopy type theory, homotopy groups play a fundamental role in that they capture information about iterated identity types. Unfortunately, even in classical topology, the computation of homotopy groups is a notoriously difficult problem. Nevertheless, topologists have come up with a variety of powerful tools for attacking this problem, and one of the most basic tools is the Hurewicz theorem. In most cases, it is much easier to compute homology groups than homotopy groups, and so one can use the isomorphism from right to left (with $A$ taken to be the integers) to compute certain homotopy groups. Moreover, one can apply the theorem even when $X$ is not $(n-1)$-connected using the following technique. Let $X\langle n-1\rangle$ denote the fibre of the truncation map $X \to \left\|X\right\|_{n-1}$ over the image of the basepoint. Then $X\langle n-1\rangle$ is $(n-1)$-connected and $\pi_n(X\langle n-1\rangle) \cong \pi_n(X)$, so $\pi_n(X)^{ab} \cong \tilde{H}_n(X\langle n-1\rangle)$. The Serre spectral sequence can often be used to compute the required homology group.

**Techniques and main results.** We first recall that for $n \geq 1$, the $n$th homology group $\tilde{H}_n(X;A)$ of a type $X$ with coefficients in an abelian group $A$ is defined to be the colimit of a certain sequential diagram

$$\pi_{n+1}(X \wedge K(A,1)) \longrightarrow \pi_{n+2}(X \wedge K(A,2)) \longrightarrow \pi_{n+3}(X \wedge K(A,3)) \longrightarrow \cdots . \qquad (1.1)$$

Here $\wedge$ denotes the smash product and $K(A,m)$ is the Eilenberg–Mac Lane space constructed in [LF14], which is an $m$-truncated, $(m-1)$-connected, pointed type with a canonical isomorphism $\pi_m(K(A,m)) \cong A$.

We now state one of our main results, which is used to prove the Hurewicz theorem, and also has other consequences:

**Theorem S** (Corollary 2.32 and Theorem 2.38)**.** *If $X$ is a pointed, $(n-1)$-connected type $(n \geq 1)$ and $Y$ is a pointed, $(m-1)$-connected type $(m \geq 1)$, then $X \wedge Y$ is $(n+m-1)$-connected and $\pi_{n+m}(X \wedge Y)$ is the tensor product of $\pi_n(X)^{ab}$ and $\pi_m(Y)^{ab}$ in a natural way.*

Taking $Y$ to be $K(A,m)$ in this result shows that the groups appearing in the sequential diagram (1.1) are tensor products of $\pi_n(X)^{ab}$ and $A$. The proof of the Hurewicz theorem follows from showing that the induced maps are isomorphisms, which we do in Lemma 3.11. With this ingredient, we prove the Hurewicz theorem as Theorem 3.12.

In order to define the isomorphism appearing in Theorem S, we must give a bilinear map $\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)$. To do so, we define and study a more general natural map

$$\mathsf{smashing} : (X \to_\bullet Y \to_\bullet Z) \longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z))$$

for any pointed types $X$, $Y$ and $Z$ and any $n, m \geq 1$. The map we require is obtained by applying $\mathsf{smashing}$ to the natural map $X \to_\bullet Y \to_\bullet X \wedge Y$.

Constructing the map $\mathsf{smashing}$ requires some work. While it lands in group homomorphisms between (0-truncated) groups, in order to construct it, we pass through *magmas*. A magma is a (not necessarily truncated) type $M$ with a binary operation $\cdot : M \times M \to M$, with no conditions or coherence laws. As a technical trick which simplifies the formalization, we work with *weak magma morphisms*. A weak magma morphism from a magma $M$ to a magma $N$ is a map $f : M \to N$ which *merely* has the property that it respects the operations. This is sufficient for our purposes, because when $M$ and $N$ are groups, it reproduces the notion of group homomorphism. All loop spaces are magmas under path concatenation, and many natural maps involving loop spaces are weak magma morphisms. By working with magmas, we can factor the map $\mathsf{smashing}$ into simpler pieces, and still land in group homomorphisms at the end, without keeping track of higher coherences.

Proving the rest of Theorem S requires a number of results that build on work in [BDR18]. For example, Lemma 2.15 and Theorem 2.19 are results of [BDR18], which we use to prove Proposition 2.23: for $n \geq 1$, $X$ a pointed, $(n-1)$-connected type, and $Y$ a pointed, $n$-truncated type, the map $\Omega^n : (X \to_\bullet Y) \to (\Omega^n X \to_{\mathsf{Mgm}} \Omega^n Y)$ is an equivalence. In order to prove this, we prove results about connected covers in Section 2.3.

We go on to define a natural Hurewicz homomorphism $h_n : \pi_n(X)^{ab} \otimes A \to \tilde{H}_n(X;A)$, without assuming any connectivity hypothesis on $X$, and show that it is unique up to a sign among such natural transformations that give isomorphisms for $X \equiv S^n$ and $A \equiv \mathbb{Z}$ (Theorem 3.16).

**Homology.** The theory of homology in homotopy type theory is currently limited by the absence of some important tools and facts that would make it easier to compute. For example, we don't have complete proofs that homology satisfies the Eilenberg–Steenrod axioms, although partial work was done by [Gra17]. The Serre spectral sequence for homology has not been formalized, but high level arguments can be found in [Doo18] and it is expected that techniques similar to those used for cohomology will go through. We are also missing the fact that the homology of a cellular space can be computed cellularly (which is done for cohomology in [BHF18]), the universal coefficient theorem, and the relationship between homology and localization (developed in homotopy type theory in [CORS20] and [Sco20]).

**Structure of the paper.** Section 2 contains our work on smash products and tensor products. After listing our conventions in Section 2.1, we give the basic theory of 1-coherent categories in Section 2.2. We use this theory to express and reason about natural transformations, and we make use of the Yoneda lemma in this setting. In Section 2.3 we study connected covers. Section 2.4 introduces magmas and weak magma morphisms, and proves a variety of results about loop spaces, including Proposition 2.23, mentioned above. We also define the map $\mathsf{smashing}$ in this section. We introduce smash products in Section 2.5 and prove the connectivity part of Theorem S here. Section 2.6 is a short section that defines abelianization and gives a particularly efficient construction of the abelianization of a group as a higher inductive type. In Section 2.7, we define tensor products of abelian groups and prove the second part of Theorem S. Section 2.8 proves results about smash products, truncation and suspension that are needed in Section 3.

Section 3 applies the results of Section 2 to homology, leading up to the Hurewicz theorem and its consequences. In Section 3.1, we define prespectra and their stable homotopy groups, and use this to define homology. The Hurewicz theorem is proved in Section 3.2, and we describe the Hurewicz homomorphism and its uniqueness up to sign in Section 3.3. In Section 3.4, we give some applications of our main results.

**Formalization.** Formalization of these results is in progress, with help from Ali Caglayan, using the Coq HoTT library [HoTT]. The current status can be seen at [GH], where the `README.md` file explains where results from the paper can be found. Currently, we have formalized much of Section 2 but none of Section 3. In Section 2, the only substantial result that is missing is Theorem 2.38. Also missing are Theorem 2.28 and the naturality of many of the maps defined in this section. In our formalization, we take as axioms several results that have been formalized in other proof assistants.

## 2. Smash products and tensor products

In this section, we give a variety of results about loop spaces, magmas, smash products and tensor products, including the proof of Theorem S. None of the results in this section depend on the definition of homology, but these results are used in the next section to prove the Hurewicz theorem.

### 2.1. Background and conventions.

We follow the conventions and notation used in [Uni13]. We assume we have a univalent universe $\mathcal{U}$ closed under higher inductive types (HITs) and contained in another universe $\mathcal{U}'$. In fact, the higher inductive types we use can all be described using pushouts and truncations. By convention, all types live in the lower universe $\mathcal{U}$, unless explicitly stated. We implicitly use function extensionality for $\mathcal{U}$ throughout.

A **pointed type** is a type $X$ and a choice of $x_0 : X$, and the type of pointed types is denoted $\mathcal{U}_\bullet :\equiv \sum_{(X:\mathcal{U})} X$. We often keep the choice of basepoint implicit. A **pointed map** between pointed types $X$ and $Y$ is a map $f : X \to Y$ and a path $p : f(x_0) = y_0$. The type of pointed maps is denoted $X \to_\bullet Y :\equiv \sum_{(f:X\to Y)} f(x_0) = y_0$.

We frequently make use of functions of type $X \to Y \to Z$, and remind the reader that this associates as $X \to (Y \to Z)$, which is the curried form of a function $X \times Y \to Z$.

In the paper, we define the sum $m + n$ of natural numbers by induction on $n$, so that $m + 1$ is the successor of $m$. In the HoTT library, the other convention is used, so to translate between the paper and the formalization, one must change $m + n$ to $n + m$ everywhere.

2.2. **1-coherent categories.** In this section, we briefly discuss the notion of 1-coherent category, which we use to express that various constructions are natural. The definitions generalize those of [Doo18, Section 4.3.1], which deals with the 1-coherent category of pointed types, except that our hom types are unpointed. A more general notion of wild category has been formalized in the HoTT library [HoTT] by Ali Caglayan, tslil clingman, Floris van Doorn, Morgan Opie, Mike Shulman and Emily Riehl.

Recall that $\mathcal{U}'$ is a universe such that $\mathcal{U} : \mathcal{U}'$.

**Definition 2.1.** A **1-coherent category** $C$ consists of a type $C_0 : \mathcal{U}'$, a map $\hom_C : C_0 \to C_0 \to \mathcal{U}$, maps

$$\mathsf{id} : \prod_{a:C_0} \hom_C(a,a),$$
$$- \circ - : \prod_{a,b,c:C_0} \hom_C(b,c) \to \hom_C(a,b) \to \hom_C(a,c),$$

and equalities

$$\mathsf{unitl} : \prod_{a,b:C_0} \prod_{f:\hom_C(a,b)} \mathsf{id}_b \circ f = f$$
$$\mathsf{unitr} : \prod_{a,b:C_0} \prod_{f:\hom_C(a,b)} f \circ \mathsf{id}_a = f$$
$$\mathsf{assoc} : \prod_{a,b,c,d:C_0} \prod_{f:\hom_C(a,b)} \prod_{g:\hom_C(b,c)} \prod_{h:\hom_C(c,d)} (h \circ g) \circ f = h \circ (g \circ f),$$

witnessing left and right unitality and associativity, respectively. We do not assume coherence laws or that any of the types are truncated.

If $C$ is a 1-coherent category, the elements of $C_0$ are called **objects** and, for objects $a, b : C_0$, the elements of $\hom_C(a,b)$ are called **morphisms** from $a$ to $b$.

The wild 1-categories considered in [HoTT] allow 2-cells to be specified, which are then used in place of the identity types in the above equalities. For simplicity, we use the identity types.

*Example* 2.2. There is a 1-coherent category $\mathsf{U}$ of types, with $\mathsf{U}_0 :\equiv \mathcal{U}$ and $\hom_{\mathsf{U}}(X,Y) :\equiv X \to Y$ for every pair of types $X, Y : \mathcal{U}$. Identity morphisms, composition, unitalities, and associativity all work in the expected way.

*Example* 2.3. There is a 1-coherent category $\mathsf{Grp}$ of groups whose objects are the set-level groups, that is, 0-truncated types equipped with an associative binary operation, a unit and inverses. The morphisms are standard group homomorphisms.

Similarly, there is 1-coherent category $\mathsf{Ab}$ of abelian groups.

*Example* 2.4. Any precategory in the sense of [Uni13, Definition 9.1.1] gives rise to a 1-coherent category, simply by forgetting that its hom types are sets. Moreover, the notions of isomorphism, functor, and natural transformation given in [Uni13, Section 9] are equivalent to the notions we give in this section, in the case of precategories.

Many constructions one can carry out with categories are easy to extend to 1-coherent categories. We mention two that are particularly important for us. Given a 1-coherent category $C$, we can form the **opposite** 1-coherent category $C^{\mathsf{op}}$ by letting the type of objects of $C^{\mathsf{op}}$ be $C_0$, and $\hom_{C^{\mathsf{op}}}(a,b) :\equiv \hom_C(b,a)$ for all $a,b : C_0$. The rest of the structure is straightforward to define.

Given 1-coherent categories $C$ and $D$ one can form a **product** 1-coherent category, denoted by $C \times D$. The underlying type of $C \times D$ is simply $C_0 \times D_0$, and $\hom_{C\times D}((c,d),(c',d')) :\equiv \hom_C(c,c') \times \hom_D(d,d')$. The rest of the structure is again straightforward to define.

**Definition 2.5.** Let $C$ be a 1-coherent category, $a,b : C_0$, and $f : \hom_C(a,b)$. An **isomorphism structure** for $f$ is given by morphisms $g,h : \hom_C(b,a)$ together with paths $l : g \circ f = \mathsf{id}_a$ and $r : f \circ h = \mathsf{id}_b$.

In many cases, such as in the 1-coherent category $\mathsf{U}$, being an isomorphism is a mere property of a morphism. The wild 1-categories considered in [HoTT] allow biinvertibility to be replaced by more general notions of isomorphism.

**Definition 2.6.** A **1-coherent functor** $F$ between 1-coherent categories $C$ and $D$, usually denoted by $F : C \to D$, consists of a map $F_0 : C_0 \to D_0$, a map

$$F_1 : \prod_{a,b:C_0} \hom_C(a,b) \to \hom_D(F_0(a),F_0(b)),$$

and equalities

$$F_{\mathsf{id}} : \prod_{a:C_0} F(\mathsf{id}_a) = \mathsf{id}_{F(a)},$$
$$F_\circ : \prod_{a,b,c:C_0} \prod_{f:\hom_C(a,b)} \prod_{g:\hom_C(b,c)} F_1(g) \circ F_1(f) = F(g \circ f),$$

witnessing the functoriality of $F$.

*Example* 2.7. For a 1-coherent category $C$, we can define a 1-coherent corepresentable functor $\mathcal{Y}^a : C \to \mathsf{U}$. On objects we have $\mathcal{Y}^a_0(b) :\equiv \hom_C(a,b)$. The action on morphisms is defined as $\mathcal{Y}^a_1(f) :\equiv \lambda g.f \circ g : \hom_C(a,b) \to \hom_C(a,c)$ for $f : \hom_C(b,c)$. The witnesses of functoriality, that is $\mathcal{Y}^a_{\mathsf{id}}$ and $\mathcal{Y}^a_\circ$, are defined using the equalities $\mathsf{unitl}$ and $\mathsf{assoc}$ of $C$, respectively.

**Definition 2.8.** Let $C$ and $D$ be 1-coherent categories and let $F,G : C \to D$ be 1-coherent functors. A **1-coherent natural transformation** $\alpha$ from $F$ to $G$, usually denoted by $\alpha : F \to G$, consists of a map

$$\alpha_0 : \prod_{a:C_0} \hom_D(F(a),G(a)),$$

and equalities

$$\alpha_1 : \prod_{a,b:C_0} \prod_{f:\hom_C(a,b)} \alpha_0(b) \circ F_1(f) = G_1(f) \circ \alpha_0(a).$$

**Definition 2.9.** Let $\alpha : F \to G$ be a 1-coherent natural transformation between 1-coherent functors $F, G : C \to D$, for $C$ and $D$ 1-coherent categories. An **isomorphism structure** for $\alpha$ is given by an isomorphism structure for each of its components. A **natural isomorphism** is given by a natural transformation together with an isomorphism structure.

The following lemma is straightforward.

**Lemma 2.10.** *Let $C$ and $D$ be 1-coherent categories, $F, G, H : C \to D$ 1-coherent functors, and $\alpha : F \to G$ and $\beta : G \to H$ 1-coherent natural transformations. Then, by defining $(\beta \circ \alpha)(c) :\equiv \beta(c) \circ \alpha(c)$ and the naturality squares by composing the naturality squares of $\alpha$ and $\beta$, one obtains a natural transformation $\beta \circ \alpha : F \to H$. Moreover, if both $\alpha$ and $\beta$ are natural isomorphisms, so is $\beta \circ \alpha$.* □

The following is a 1-coherent version of the fact that the Yoneda functor is an embedding.

**Proposition 2.11** ([HoTT])**.** *Let $C$ be a 1-coherent category and let $a, b : C_0$. Assume given a 1-coherent natural isomorphism $\alpha : \mathcal{Y}^b \to \mathcal{Y}^a$. Then $i :\equiv \alpha_0(b)(\mathsf{id}_b) : \hom_C(a, b)$ is part of an isomorphism between $a$ and $b$, and it satisfies, for every $c : C_0$,*

$$\alpha_0(c) = \lambda g.g \circ i$$

*as maps $\hom_C(b, c) \to \hom_C(a, c)$.* □

The proof is the same as the usual proof, and has been formalized in the HoTT library [HoTT]. Note that we are not claiming that the naturality proofs for $\alpha$ can be recovered using the associativity of composition.

2.3. **Connected covers.** In order to generalize a result of Buchholtz, van Doorn and Rijke (see Theorem 2.19) to the case where $Y$ has no connectivity assumption, we prove some results about connected covers. In this section, we fix $n \geq -1$.

**Definition 2.12.** A type $X$ is ***n*-connected** if $\left\|X\right\|_n$ is contractible.

For $X$ pointed, it is equivalent to require that $\pi_i(X)$ be trivial for all $i \leq n$. Every pointed type is $(-1)$-connected.

**Definition 2.13.** Let $X$ be a pointed type. The ***n*-connected cover** $X\langle n\rangle$ of $X$ is defined to be the fibre of the pointed map $X \to_\bullet \left\|X\right\|_n$.

Note that $X\langle n\rangle$ is indeed $n$-connected and that we have a canonical pointed map $i : X\langle n\rangle \to_\bullet X$ which induces an equivalence on the homotopy groups $\pi_k$ for $k > n$. In fact, this map has a stronger universal property:

**Definition 2.14.** A pointed map $f : X \to_\bullet Y$ is an **$\langle n\rangle$-equivalence** if for any pointed, $n$-connected type $Z$, post-composition by $f$ gives an equivalence

$$(Z \to_\bullet X) \xrightarrow{\sim} (Z \to_\bullet Y).$$

**Lemma 2.15** ([BDR18, Lemma 6.2])**.** *Let $X$ be a pointed type. Then the map $i : X\langle n\rangle \to_\bullet X$ is an $\langle n\rangle$-equivalence.* □

It follows that the operation sending $X$ to $X\langle n\rangle$ is functorial in a unique way making $i : X\langle n\rangle \to_\bullet X$ natural, and that a map $f$ is an $\langle n\rangle$-equivalence if and only if $f\langle n\rangle$ is an equivalence.

Note that there is a 1-coherent category with objects all pointed types and morphisms given by pointed functions. We denote this 1-coherent category by $\mathsf{U}_\bullet$. There are 1-coherent functors $\Sigma, \Omega : \mathsf{U}_\bullet \to \mathsf{U}_\bullet$ forming a 1-coherent adjunction, in the following sense.

**Lemma 2.16** ([Uni13, Lemma 6.5.4]). *Let $X$ and $Y$ be pointed types. There is an equivalence*

$$(\Sigma X \to_\bullet Y) \simeq (X \to_\bullet \Omega Y),$$

*natural in $X$ and $Y$. Here, we are interpreting $(\Sigma(-) \to_\bullet -)$ and $(- \to_\bullet \Omega(-))$ as 1-coherent functors $\mathsf{U}_\bullet^{\mathsf{op}} \times \mathsf{U}_\bullet \to \mathsf{U}$.* □

The naturality is not proven in [Uni13], but is proven in the HoTT library [HoTT].

The following two facts will be used in Proposition 2.23.

**Proposition 2.17.** *Let $f : X \to_\bullet Y$ be a pointed map. If $f$ is an $\langle n+1\rangle$-equivalence, then $\Omega f$ is an $\langle n\rangle$-equivalence.*

*Proof.* Let $A$ be an $n$-connected, pointed type. By naturality of the adjunction between suspension and loops (Lemma 2.16), we have a commutative square

$$\begin{array}{ccc} (\Sigma A \to_\bullet X) & \xrightarrow{f\circ -} & (\Sigma A \to_\bullet Y) \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ (A \to_\bullet \Omega X) & \xrightarrow{\Omega f \circ -} & (A \to_\bullet \Omega Y) \end{array}$$

in which the vertical maps are equivalences. Since the suspension of an $n$-connected type is $(n+1)$-connected, the top map is also an equivalence. Therefore, the bottom map is an equivalence, as required. □

**Proposition 2.18.** *Let $f : X \to_\bullet Y$ be a pointed map. If $f$ is a $\langle -1\rangle$-equivalence, then $f$ is an equivalence.*

*Proof.* Since $S^0$ is $(-1)$-connected, we know that $f$ induces an equivalence $(S^0 \to_\bullet X) \to_\bullet (S^0 \to_\bullet Y)$. Moreover, $(S^0 \to_\bullet Z)$ is equivalent to $Z$ for any pointed type $Z$, and this equivalence is natural. It follows that $f$ is an equivalence. □

This also follows from the facts that $Z\langle -1\rangle \to_\bullet Z$ is an equivalence for any pointed $Z$, and that $f\langle -1\rangle$ is an equivalence.

2.4. **Loop spaces and magmas.** In this section, we study loop spaces and the natural magma structures that they carry and define the map $\mathsf{smashing}$ that plays an important role in this paper. We begin by generalizing the following result of Buchholtz, van Doorn and Rijke.

**Theorem 2.19** ([BDR18, Theorem 5.1]). *Let $n \geq 1$. For $X$ and $Y$ pointed, $(n-1)$-connected, $n$-truncated types, the map*

$$\Omega^n : (X \to_\bullet Y) \longrightarrow_\bullet (\Omega^n X \to_{\mathsf{Grp}} \Omega^n Y).$$

*is an equivalence.* □

In order to state our generalization, we introduce the notion of magma.

**Definition 2.20.** A **magma** is given by a type $X$ together with an operation $\cdot_X : X \times X \to X$. A **map of magmas** is given by a map $f : X \to Y$ between the underlying types that merely respects the operations. More formally, we define

$$X \to_{\mathsf{Mgm}} Y :\equiv \sum_{(f:X\to Y)} \left\| \prod_{(x,x':X)} f(x \cdot_X x') = f(x) \cdot_Y f(x') \right\|_{-1}.$$

An **equivalence of magmas** is a map of magmas whose underlying map is an equivalence. We write $X \simeq_{\mathsf{Mgm}} Y$ for the type of magma equivalences from $X$ to $Y$. Magmas form a 1-coherent category that we denote $\mathsf{Mgm}$. We will omit the subscript on the operation $\cdot$ when it is clear from context.

The propositional truncation in the definition of magma map is a technical trick to simplify the formalization. With our definition, the type of equalities between magma maps is equivalent to the type of equalities between the underlying maps. All of our results should go through without the truncation, but omitting it leads to path algebra that is not needed in order to get our later results. The maps we are considering should be called "weak magma maps," but since they are the only maps we use, we simply call them "magma maps" in this paper.

**Definition 2.21.** A **pointed magma** is a magma $X$ with a chosen point $x_0 : X$ and an equality $x_0 \cdot x_0 = x_0$. A **map of pointed magmas** is a pointed map $f : X \to_\bullet Y$ whose underlying map $f : X \to Y$ is a map of magmas. We write $X \to_{\mathsf{Mgm}_\bullet} Y$ for the type of pointed magma maps. An **equivalence of pointed magmas** is a map of pointed magmas whose underlying map is an equivalence. We write $X \simeq_{\mathsf{Mgm}_\bullet} Y$ for the type of pointed magma equivalences. Pointed magmas form a 1-coherent category that we denote $\mathsf{Mgm}_\bullet$.

There are no propositional truncations in the above definition, except for the one in the definition of magma map.

*Remark* 2.22. The loop space $\Omega X$ is a pointed magma for any pointed type $X$, with path concatenation as the operation, reflexivity as the basepoint, and a higher reflexivity as the proof that the basepoint is idempotent. There is a natural map $\Omega : (X \to_\bullet Y) \to_\bullet (\Omega X \to_{\mathsf{Mgm}_\bullet} \Omega Y)$, which can be iterated. Any magma map $\Omega X \to_{\mathsf{Mgm}} \Omega Y$ induces a group homomorphism $\pi_1(X) \to_{\mathsf{Grp}} \pi_1(Y)$. Also note that for groups $G$ and $H$, $(G \to_{\mathsf{Grp}} H) \simeq (G \to_{\mathsf{Mgm}} H)$, where we write $G \to_{\mathsf{Grp}} H$ for the type of group homomorphisms. (We assume that all groups have an underlying type that is a set, which means that the propositional truncation can be removed.)

When $X$ is a pointed magma and $G$ is a group, every magma map $X \to_{\mathsf{Mgm}} G$ can be made pointed in a unique way, so the forgetful map $(X \to_{\mathsf{Mgm}_\bullet} G) \to (X \to_{\mathsf{Mgm}} G)$ is an equivalence.

When $A$ is a pointed type and $X$ is a pointed magma, the type $A \to_\bullet X$ of pointed maps is a pointed magma under the pointwise operation. The requirement that the basepoint $x_0 : X$ be idempotent ensures that for $f, g : A \to_\bullet X$, $f \cdot g$ is again pointed: $(f \cdot g)(a_0) \equiv f(a_0) \cdot g(a_0) = x_0 \cdot x_0 = x_0$.

Similarly, when $Y$ is a pointed magma and $Z$ is a pointed type, the type $Y \to_{\mathsf{Mgm}_\bullet} \Omega^2 Z$ of pointed magma maps and the type $Y \to_{\mathsf{Mgm}} \Omega^2 Z$ of all magma maps are pointed magmas under the pointwise operation. This uses that path composition in the double loop space is commutative (by Eckmann–Hilton) and associative. (More precisely, we only use that the operation is merely commutative and merely associative, which will be convenient in Definition 2.26.)

With this background, we can now state our first generalization of Theorem 2.19.

**Proposition 2.23.** *Let $n \geq 1$, let $X$ be a pointed, $(n-1)$-connected type, and let $Y$ be a pointed, $n$-truncated type. Then the map*

$$\Omega^n : (X \to_\bullet Y) \xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}} \Omega^n Y),$$

*is an equivalence, natural in $X$ and $Y$. Similarly,*

$$\Omega^n : (X \to_\bullet Y) \xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}_\bullet} \Omega^n Y),$$

*is a natural equivalence.*

*Proof.* Since $\Omega^n Y$ is a group, the second equivalence follows from the first, using Remark 2.22, so we focus on the first one. By the functoriality of $\Omega^n$, the diagram

$$\begin{array}{ccc}
(X \to_\bullet Y) & \longrightarrow & (\Omega^n X \to_{\mathsf{Mgm}} \Omega^n Y) \\
\uparrow & & \uparrow \\
(\left\|X\right\|_n \to_\bullet Y) & \longrightarrow & (\Omega^n (\left\|X\right\|_n) \to_{\mathsf{Mgm}} \Omega^n Y) \\
\uparrow & & \uparrow \\
(\left\|X\right\|_n \to_\bullet Y\langle n-1\rangle) & \longrightarrow & (\Omega^n (\left\|X\right\|_n) \to_{\mathsf{Mgm}} \Omega^n (Y\langle n-1\rangle))
\end{array}$$

commutes, where the vertical maps are induced by the maps $|-|_n : X \to_\bullet \left\|X\right\|_n$ and $i : Y\langle n-1\rangle \to_\bullet Y$. The vertical maps on the left are equivalences by the universal properties of truncations and of connected covers.

To see that the upper vertical map on the right is an equivalence, let $f$ denote the map $\Omega^n(|-|_n) : \Omega^n X \to_\bullet \Omega^n(\left\|X\right\|_n)$. This map is 0-connected, since $|-|_n$ is $n$-connected and $\Omega$ decreases connectivity. Since $\Omega^n Y$ is a set, it follows that $f$ induces an equivalence $(\Omega^n(\left\|X\right\|_n) \to \Omega^n Y) \to (\Omega^n(X) \to \Omega^n Y)$. Given $g : \Omega^n(\left\|X\right\|_n) \to \Omega^n Y$, we need to show that $g$ merely preserves the magma structures if and only if $g o f$ merely preserves the magma structures. The map $f$ induces an equivalence

$$\left( \prod_{a,b:\Omega^n(\|X\|_n)} g(a \cdot b) = g(a) \cdot g(b) \right) \simeq \left( \prod_{a,b:\Omega^n(X)} g(f(a) \cdot f(b)) = g(f(a)) \cdot g(f(b)) \right),$$

since $f$ is 0-connected and the identity types are sets (in fact, propositions). Note that $f$, being defined using the functoriality of $\Omega$, preserves the concatenation operation (without any propositional truncation). It follows that the type on the right is equivalent to the type of proofs that $g \circ f$ preserves the magma structure. Therefore, the propositional truncations are also equivalent, so $f$ induces an equivalence on the types of magma maps.

The lower vertical map on the right is an equivalence since $\Omega^n(i)$ is an equivalence of magmas: it is certainly a map of magmas, and the fact that it is an equivalence follows from Propositions 2.17 and 2.18.

The bottom horizontal map is an equivalence by Theorem 2.19, and so the top horizontal map is an equivalence, as required.

The fact that $\Omega^n$ is natural in $X$ and $Y$ follows from the functoriality of $\Omega^n$ as an operation from pointed maps to magma maps, which is straightforward to check. □

Our next goal is to define the map `smashing`, using the following lemmas.

**Lemma 2.24.** *Let $n \geq 1$ and let $Y$ and $Z$ be pointed types. Then there is an equivalence of pointed magmas*

$$\Omega^n(Y \to_\bullet Z) \simeq_{\mathsf{Mgm}_\bullet} (Y \to_\bullet \Omega^n Z)\,,$$

*natural in $Y$ and $Z$. Here we are regarding $\Omega^n(- \to_\bullet -)$ and $- \to_\bullet \Omega^n(-)$ as 1-coherent functors $\mathsf{U}_\bullet \times \mathsf{U}_\bullet \to \mathsf{Mgm}_\bullet$.*

On the right-hand-side, we are using the pointwise magma structure described in Remark 2.22.

*Proof.* We prove this for $n = 1$, and then iterate, using that the functor $\Omega$ sends pointed equivalences to equivalences of pointed magmas.

In order to prove that our equivalence respects the magma structures, it is best to generalize: for $f, g : Y \to_\bullet Z$ we define an equivalence

$$\varphi : (f = g) \xrightarrow{\sim} \sum_{K : f \sim g} K(y_0) = f_0 \cdot g_0^{-1}.$$

Here $K$ is a homotopy, $y_0$ is the basepoint of $Y$, and $f_0$ and $g_0$ are the paths witnessing that $f$ and $g$ are pointed. This equivalence is a variant of the standard result that equalities of pointed maps are equivalent to pointed homotopies; the particular choice of the right-hand-side means that when $f$ and $g$ are the constant map $Y \to_\bullet Z$ pointed by $\mathsf{refl}$, we obtain a pointed equivalence

$$\Omega(Y \to_\bullet Z) \simeq_\bullet (Y \to_\bullet \Omega Z)\,.$$

Our pointed homotopies can be composed, and we show that $\varphi$ sends composition of paths to composition of homotopies by first doing induction on the paths to reduce the goal to

$$\varphi(\mathsf{refl}) = \varphi(\mathsf{refl}) \cdot \varphi(\mathsf{refl})$$

and then using path induction to assume that $f_0$ is $\mathsf{refl}$. We conclude that

$$\Omega(Y \to_\bullet Z) \simeq_{\mathsf{Mgm}_\bullet} (Y \to_\bullet \Omega Z)\,.$$

To prove naturality in $Y$, consider a pointed map $h : Y \to_\bullet Y'$. We must show that the following square commutes:

$$\begin{array}{ccc} \Omega(Y' \to_\bullet Z) & \xrightarrow{\Omega(-\circ h)} & \Omega(Y \to_\bullet Z) \\ \varphi \downarrow & & \downarrow \varphi \\ (Y' \to_\bullet \Omega Z) & \xrightarrow[-\circ h]{} & (Y \to_\bullet \Omega Z). \end{array}$$

By path induction, we can assume that $h$ is strictly pointed, i.e., that the given path $h_0 : h(y_0) = y_0'$ is reflexivity. In this case, writing $c : Y \to_\bullet Z$ and $c' : Y' \to_\bullet Z$ for the constant maps, we have that $c' \circ h$ and $c$ are definitionally equal as pointed maps. Therefore, the corners and vertical maps in the required square are definitionally equal to those in the square

$$\begin{array}{ccc} c' = c' & \xrightarrow{\mathsf{ap}_{-\circ h}} & c' \circ h = c' \circ h \\ \varphi \downarrow & & \downarrow \varphi \\ c' \sim_\bullet c' & \xrightarrow{\mathsf{wh}_h} & c' \circ h \sim_\bullet c' \circ h, \end{array}$$

where $\sim_\bullet$ denotes the type of pointed homotopies defined above, and $\mathsf{wh}_h$ denotes prewhiskering with $h$. One can check that the horizontal arrows are homotopic to those in the required square, so it remains to show that the new square commutes. To show this, one generalizes from $c' = c'$ to $f = g$, in which case the commutativity follows by path induction.

The proof of naturality in $Z$ is very similar. Since both naturalities have been formalized, we give no further details. □

**Lemma 2.25.** *Let $n, m \geq 1$ and let $Y$ and $Z$ be pointed types. The action of $\Omega^m$ on maps gives a pointed magma map*

$$(Y \to_\bullet \Omega^n Z) \longrightarrow_{\mathsf{Mgm}_\bullet} (\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega^n Z).$$

*Moreover, the forgetful maps*

$$(\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega^n Z) \longrightarrow_{\mathsf{Mgm}_\bullet} (\Omega^m Y \to_{\mathsf{Mgm}} \Omega^m \Omega^n Z)$$

*and*

$$(\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega^n Z) \longrightarrow_{\mathsf{Mgm}_\bullet} (\Omega^m Y \to_\bullet \Omega^m \Omega^n Z)$$

*are also pointed magma maps. In all cases, we are using the pointwise magma structure described in Remark 2.22. These maps are all natural.*

*Proof.* That the forgetful maps are natural pointed magma maps is straightforward, so we focus on the first map. By replacing $Z$ with $\Omega^{n-1} Z$, we can assume that $n = 1$. To prove that $\Omega^m$ is a natural pointed magma map, we induct on $m$. For the inductive step, we define $\Omega^{m+1}$ to be the composite

$$\begin{aligned}(Y \to_\bullet \Omega Z) &\xrightarrow{\Omega^m} (\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega Z)\\ &\longrightarrow (\Omega^m Y \to_\bullet \Omega^m \Omega Z)\\ &\xrightarrow{\Omega} (\Omega^{m+1} Y \to_{\mathsf{Mgm}_\bullet} \Omega^{m+1} \Omega Z)\end{aligned}$$

so that the claim follows from the inductive hypothesis, the fact that the middle forgetful map is a natural pointed magma map, and the $m = 1$ case.

It remains to prove the $m = 1$ case. It is easy to see that for $f : Y \to_\bullet \Omega Z$, $\Omega f$ is a pointed magma map. Next we must show that given $f, g : Y \to_\bullet \Omega Z$, $\Omega(f \cdot g)$ and $(\Omega f) \cdot (\Omega g)$ are equal as pointed magma maps, where $\cdot$ denotes the pointwise operations. Because we are using weak magma maps, it is equivalent to show that these two maps are equal as pointed maps, or in other words that there is a pointed homotopy $\Omega(f \cdot g) \sim_\bullet (\Omega f) \cdot (\Omega g)$. The underlying homotopy involves some path algebra, and ultimately follows from the fact that horizontal and vertical composition agree in the codomain, which is a double-loop space. The pointedness of the homotopy follows by a simple path induction on the paths $f(y_0) = \mathsf{refl}$ and $g(y_0) = \mathsf{refl}$, after generalizing $f(y_0)$ and $g(y_0)$ to arbitary loops. The argument in this paragraph has been formalized.

The naturality of $\Omega$ follows from the fact that for pointed maps $h$ and $k$, $\Omega(h \circ k) = \Omega(h) \circ \Omega(k)$ as pointed maps, where again we are taking advantage of the fact that we are using weak magma maps. □

**Definition 2.26.** For pointed types $X$, $Y$, and $Z$ and natural numbers $n, m \geq 1$, we have maps:

$$\begin{aligned}(X \to_\bullet Y \to_\bullet Z) &\longrightarrow (\Omega^n X \to_{\mathsf{Mgm}} \Omega^n(Y \to_\bullet Z)) \\ &\xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}} (Y \to_\bullet \Omega^n Z)) \\ &\longrightarrow (\Omega^n X \to_{\mathsf{Mgm}} (\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega^n Z)) \\ &\xrightarrow{\sim} (\Omega^n X \to_{\mathsf{Mgm}} (\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^{n+m} Z)) \\ &\longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z)).\end{aligned} \tag{2.1}$$

These maps are natural in $X$, $Y$, and $Z$. The first and third arrows apply $\Omega^n$ and $\Omega^m$ to morphisms, using Lemma 2.25. The second arrow is an equivalence by Lemma 2.24. To understand the fourth arrow, write $m = k + 1$ for some $k : \mathbb{N}$. Then $\Omega^k \Omega^n Z = \Omega^{n+k} Z$ as pointed types. Applying $\Omega$ on the outside, we see that $\Omega^m \Omega^n Z = \Omega^{n+m} Z$ as magmas. Since the magma structure on the set of magma maps only uses that the iterated loop space is *merely* commutative and *merely* associative, we can conclude that $(\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^m \Omega^n Z) = (\Omega^m Y \to_{\mathsf{Mgm}_\bullet} \Omega^{n+m} Z)$ as magmas. From this we deduce the required equivalence. The fifth arrow applies 0-truncation on the inside and then on the outside. Let

$$\mathsf{smashing} : (X \to_\bullet Y \to_\bullet Z) \longrightarrow (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(Z))$$

denote the composite.

The map $\mathsf{smashing}$ corresponds to the following construction in topology, which uses the smash product from the next section. Given a map $f : X \to_\bullet Y \to_\bullet Z$ and homotopy classes $\alpha : \pi_n(X)$ and $\beta : \pi_m(Y)$, one can smash representatives of the homotopy classes together to get an element $\alpha \wedge \beta : \pi_{n+m}(X \wedge Y)$. The adjoint $X \wedge Y \to_\bullet Z$ of $f$ then induces a map taking this to an element of $\pi_{n+m}(Z)$ which (up to sign) is $\mathsf{smashing}(f, \alpha, \beta)$. This correspondence motivates the name.

Since we'll use it several times, we quote the following result from [BDR18].

**Lemma 2.27** ([BDR18, Corollary 4.3])**.** *Let $m \geq 0$ and $n \geq -1$. If $Y$ is a pointed, $(m-1)$-connected type and $Z$ is a pointed, $(n+m)$-truncated type, then the type $Y \to_\bullet Z$ is $n$-truncated.* □

The last result in this section plays an important role in our proof, and can be thought of as a generalization of Theorem 2.19 to functions with two arguments.

**Theorem 2.28.** *Let $n, m \geq 1$. If $X$ is a pointed $(n-1)$-connected type, $Y$ is a pointed $(m-1)$-connected type, and $Z$ is a pointed $(n+m)$-truncated type, then the map* $\mathsf{smashing}$ *is an equivalence.*

*Proof.* The first arrow in Eq. (2.1) is an equivalence by Lemma 2.27 and Proposition 2.23. The third arrow is an equivalence by Proposition 2.23. To show that the fifth arrow is an equivalence, one uses the same methods as in the proof of Proposition 2.23, using that $\Omega^{n+m} Z$ is a set. □

2.5. **The connectivity of smash products.** We recall some basic facts about smash products, and then prove a result about their connectivity.

**Definition 2.29.** For pointed types $X$ and $Y$, the **smash product** $X \wedge Y$ is defined to be the higher inductive type with constructors:

- $\mathsf{sm} : X \times Y \to X \wedge Y$.
- $\mathsf{auxl} : X \wedge Y$.
- $\mathsf{auxr} : X \wedge Y$.
- $\mathsf{gluel} : \prod_{(y:Y)} \mathsf{sm}(x_0, y) = \mathsf{auxl}$.
- $\mathsf{gluer} : \prod_{(x:X)} \mathsf{sm}(x, y_0) = \mathsf{auxr}$.

The smash product is pointed by $\mathsf{sm}(x_0, y_0)$. It has the expected induction principle.

It is straightforward to see that the smash product is a functor. That is, given pointed maps $f : X \to_\bullet X'$ and $g : Y \to_\bullet Y'$ between pointed types, there is a pointed map $f \wedge g : X \wedge Y \to_\bullet X' \wedge Y'$ defined by induction on the smash product in the evident way, and this operation respects identity maps and composition.

Given pointed types $X$ and $Y$, the constructors of the smash product $X \wedge Y$ combine to give a map $X \to_\bullet (Y \to_\bullet X \wedge Y)$, which we now describe.

**Definition 2.30.** Let $X, Y : \mathcal{U}_\bullet$. Currying the constructor $\mathsf{sm}$, we get a map $X \to (Y \to X \wedge Y)$. Using the constructor $\mathsf{gluer}$ twice, this map lifts to a map $X \to (Y \to_\bullet X \wedge Y)$. Similarly, using $\mathsf{gluel}$, this last map lifts to a map $\mathsf{sm}_\bullet : X \to_\bullet (Y \to_\bullet X \wedge Y)$.

The following adjunction between pointed maps and smash products is fundamental to our work.

**Lemma 2.31** ([Doo18, Theorem 4.3.28]). *Let $X$, $Y$, and $Z$ be pointed types. The map*

$$(X \wedge Y \to_\bullet Z) \longrightarrow_\bullet (X \to_\bullet (Y \to_\bullet Z)),$$

*induced by precomposition with $\mathsf{sm}_\bullet$ is a pointed equivalence, natural in $X$, $Y$, and $Z$. Here, we are interpreting $(- \wedge - \to_\bullet -)$ and $(- \to_\bullet (- \to_\bullet -))$ as 1-coherent functors $\mathsf{U}_\bullet^{\mathsf{op}} \times \mathsf{U}_\bullet^{\mathsf{op}} \times \mathsf{U}_\bullet \to \mathsf{U}_\bullet$.* □

Note that, by construction, $\mathsf{sm}_\bullet : X \to_\bullet Y \to_\bullet X \wedge Y$ is the adjunct of the identity map $X \wedge Y \to_\bullet X \wedge Y$.

In the form stated here, Lemma 2.31 has been formalized by [Doo18]. A stronger statement, which roughly involves regarding the category $\mathsf{U}_\bullet$ as being *enriched* over $\mathsf{U}_\bullet$, has not yet been proven, but we do not use this stronger form.

We now give a bound on the connectivity of smash products, proving the first part of Theorem S from the Introduction.

**Corollary 2.32.** *Let $n, m \geq 0$, let $X$ be a pointed, $(n-1)$-connected type, and let $Y$ be a pointed, $(m-1)$-connected type. Then $X \wedge Y$ is $(n+m-1)$-connected.*

*Proof.* It is enough to show that the truncation map $X \wedge Y \to \left\|X \wedge Y\right\|_{n+m-1}$ is nullhomotopic. Since the truncation map is pointed, this follows from the following more general fact: for any pointed, $(n+m-1)$-truncated type $Z$, the type $X \wedge Y \to_\bullet Z$ is contractible. Indeed, by Lemma 2.31, we have $(X \wedge Y \to_\bullet Z) \simeq (X \to_\bullet Y \to_\bullet Z)$. By Lemma 2.27, the type $Y \to_\bullet Z$ is $(n-1)$-truncated. Therefore, using Lemma 2.27 again, we see that the type $X \to_\bullet Y \to_\bullet Z$ is $(-1)$-truncated, and thus contractible, since any pointed mapping space is inhabited. □

2.6. **Abelianization.** In this section, we introduce the notion of abelianization, and give an efficient construction of the abelianization of a group.

**Definition 2.33.** Given a group $G$, an **abelianization** of $G$ consists of an abelian group $A$ together with a homomorphism $\eta : G \to_{\mathsf{Grp}} A$, initial among homomorphisms to abelian groups. In other words, for each abelian group $B$ and homomorphism $h : G \to_{\mathsf{Grp}} B$, the type $\sum_{(f:A\to B)} h = f \circ \eta$ is contractible.

Since the type of abelianizations of a given group is a mere proposition, we abuse notation and denote any such abelianization by $G \to G^{ab}$.

*Remark* 2.34. The existence of abelianizations can be proved in several different ways. One could mimic the classical definition, describing $G^{ab}$ as the quotient of $G$ by the subgroup generated by commutators, but this is awkward to work with constructively.

A second method that clearly works is to define $G^{ab}$ as a higher inductive type with a point constructor $\eta : G \to G^{ab}$, a point constructor giving $G^{ab}$ an identity element, recursive point constructors giving addition and inverses in $G^{ab}$, recursive path constructors showing that the group laws hold and that the operation is abelian, a path constructor showing that $\eta$ is a homomorphism, and a recursive path constructor forcing $G^{ab}$ to be a set. While there is no doubt that this will work, it is difficult to use in practice because of the number of constructors and the fact that many of them are recursive.

A much simpler construction is as the higher inductive type with the following constructors:

- $\eta : G \to G^{ab}$.
- $\mathsf{comm} : \prod_{a,b,c:G} \eta(a \cdot (b \cdot c)) = \eta(a \cdot (c \cdot b))$.
- $\mathsf{isset} : \prod_{x,y:G^{ab}} \prod_{p,q:x=y} p = q$.

Equivalently, this is the 0-truncation of the coequalizer of the two obvious maps $G \times G \times G \to G$. Using either description, it is straightforward to show that $G^{ab}$ has a unique group structure making $\eta$ a group homomorphism, that this group structure is abelian, and that $\eta$ satisfies the universal property.

We don't give further details here, since we do not use the fact that abelianizations exist in this paper. (In fact, the existence of abelianizations follows indirectly from the Hurewicz theorem.) Moreover, the sketch presented here has been formalized by Ali Caglayan in the HoTT library [HoTT].

Given a group homomorphism $f : G \to_{\mathsf{Grp}} H$, there is a unique group homomorphism $f^{ab} : G^{ab} \to_{\mathsf{Grp}} H^{ab}$ making the square

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ G^{ab} & \xrightarrow{f^{ab}} & H^{ab} \end{array}$$

commute. This makes abelianization into a functor and $\eta$ into a natural transformation.

2.7. **Tensor products.** In this section, we define tensor products and use them to complete the proof of Theorem S.

Recall that for a group $G$ and an abelian group $H$, the set $G \to_{\mathsf{Grp}} H$ is an abelian group. The group operation is given by $(\varphi + \psi)(g) :\equiv \varphi(g) + \psi(g)$, and the inverse by $(-\psi)(g) :\equiv -\psi(g)$, along with the natural proofs that these are homomorphisms.

**Definition 2.35.** Given abelian groups $A, B$, a **tensor product** of $A$ and $B$ consists of an abelian group $T$ together with a map $t : A \to_{\mathsf{Grp}} B \to_{\mathsf{Grp}} T$ such that for any abelian group $C$ the map

$$t^* : (T \to_{\mathsf{Grp}} C) \longrightarrow (A \to_{\mathsf{Grp}} B \to_{\mathsf{Grp}} C)$$

given by composition with $t$ is an equivalence.

One can show in a straightforward way that tensor products exist, although we don't need this, and in fact the existence follows from Theorem 2.38. Moreover, the type of tensor products of a given pair of abelian groups is a mere proposition. We denote any such tensor product by $A \otimes B$. Given $a : A$, and $b : B$, we form the **elementary tensor** $a \otimes b : A \otimes B$ as $a \otimes b :\equiv t(a, b)$.

*Example* 2.36. Let $A : \mathsf{Ab}$. Then $A \simeq A \otimes \mathbb{Z}$, and the isomorphism is given by mapping $a : A$ to $a \otimes 1$. This follows from the fact that $\mathbb{Z}$ represents the identity, that is, $(\mathbb{Z} \to_{\mathsf{Grp}} C) \simeq_{\mathsf{Grp}} C$ for any $C : \mathsf{Ab}$, where the isomorphism is given by mapping $f : \mathbb{Z} \to_{\mathsf{Grp}} C$ to $f(1)$.

**Lemma 2.37.** *Let $A, B, C : \mathsf{Ab}$, and $\varphi, \psi : A \otimes B \to_{\mathsf{Grp}} C$. If for every $a : A$ and $b : B$ we have $\varphi(a \otimes b) = \psi(a \otimes b)$, then $\varphi = \psi$.*

*Proof.* By construction, we have $\varphi(a \otimes b) = t^*(\varphi)(a, b)$ and $\psi(a \otimes b) = t^*(\psi)(a, b)$. By assumption and function extensionality, we have $t^*(\varphi) = t^*(\psi)$, and since $t^*$ is an equivalence, we deduce that $\varphi = \psi$. □

A key step towards proving the Hurewicz theorem is constructing a map $\pi_n(X)^{ab} \otimes \pi_m(Y)^{ab} \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)$ natural in the pointed types $X$ and $Y$, and proving that this map is an equivalence under connectivity assumptions on $X$ and $Y$. Equivalently, we are looking for a map $\pi_n(X)^{ab} \to_{\mathsf{Grp}} \pi_m(Y)^{ab} \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)$ that is a tensor product under these assumptions.

In order to do this, observe that, for $G$ and $H$ groups and $A$ an abelian group, we have an equivalence

$$(G^{ab} \to_{\mathsf{Grp}} H^{ab} \to_{\mathsf{Grp}} A) \xrightarrow{\sim} (G \to_{\mathsf{Grp}} H \to_{\mathsf{Grp}} A),$$

given by precomposition with the corresponding abelianization maps. Applying $\mathsf{smashing}$ to the map $\mathsf{sm}_\bullet : X \to_\bullet Y \to_\bullet X \wedge Y$ from Definition 2.30 and using the above observation, we get a natural map

$$t_{X,Y} : \pi_n(X)^{ab} \to_{\mathsf{Grp}} \pi_m(Y)^{ab} \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y).$$

**Theorem 2.38.** *Let $n, m \geq 1$, let $X$ be a pointed, $(n-1)$-connected type, and let $Y$ be a pointed, $(m-1)$-connected type. Then the map $t_{X,Y}$ exhibits $\pi_{n+m}(X \wedge Y)$ as the tensor product of $\pi_n(X)^{ab}$ and $\pi_m(Y)^{ab}$.*

This implies in particular that tensor products of abelian groups exist.

*Proof.* Given an abelian group $C$, we must show that the map

$$t^*_{X,Y} : (\pi_{n+m}(X \wedge Y) \to_{\mathsf{Grp}} C) \longrightarrow \left(\pi_n(X)^{ab} \to_{\mathsf{Grp}} \pi_m(Y)^{ab} \to_{\mathsf{Grp}} C\right).$$

is an equivalence. The following diagram will let us show that $t^*_{X,Y}$ is homotopic to a map that is easily proven to be an equivalence. Let $h : \pi_{n+m}(X \wedge Y) \to_{\mathsf{Grp}} C$ and consider the diagram:

$$\begin{array}{ccc}
 & & (\pi_{n+m}(X \wedge Y) \to_{\mathsf{Grp}} C) \\
 & & \uparrow {\scriptstyle \sim}\ {\scriptstyle \pi_{n+m}} \\
(X \wedge Y \to_\bullet X \wedge Y) & \xrightarrow{h'_*} & (X \wedge Y \to_\bullet K(C, n+m)) \\
\downarrow {\scriptstyle \sim} & & {\scriptstyle \sim} \downarrow \\
(X \to_\bullet Y \to_\bullet X \wedge Y) & \xrightarrow{h'_*} & (X \to_\bullet Y \to_\bullet K(C, n+m)) \\
{\scriptstyle \mathsf{smashing}} \downarrow & & {\scriptstyle \sim} \downarrow {\scriptstyle \mathsf{smashing}} \\
(\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)) & \xrightarrow{h_*} & (\pi_n(X) \to_{\mathsf{Grp}} \pi_m(Y) \to_{\mathsf{Grp}} C) \\
\uparrow {\scriptstyle \sim} & & {\scriptstyle \sim} \uparrow \\
\big(\pi_n(X)^{ab} \to_{\mathsf{Grp}} \pi_m(Y)^{ab} \to_{\mathsf{Grp}} \pi_{n+m}(X \wedge Y)\big) & \xrightarrow{h_*} & \big(\pi_n(X)^{ab} \to_{\mathsf{Grp}} \pi_m(Y)^{ab} \to_{\mathsf{Grp}} C\big)
\end{array}$$

We explain the diagram. The right-hand vertical arrow at the top is an equivalence by Corollary 2.32 and Proposition 2.23, and also implicitly uses a chosen equivalence $e : \pi_{n+m}(K(C, n+m)) \simeq C$. The unlabeled vertical arrows bordering the first square are the adjunction from Lemma 2.31. The vertical arrows labelled $\mathsf{smashing}$ are from Definition 2.26; the right-hand one uses $e$ and is an equivalence by Theorem 2.28. The unlabeled vertical arrows at the bottom are from the universal property of abelianization. The horizontal maps labelled $h_*$ are postcomposition by $h$. The horizontal maps labelled $h'_*$ are postcomposition with the map $h' : X \wedge Y \to_\bullet K(C, n+m)$ which corresponds to $h$ under the displayed equivalence $\pi_{n+m}$. It is straightforward to check that the three squares commute.

The right-hand column is an equivalence which we will show is homotopic to $t^*_{X,Y}$. Consider the identity map $\mathsf{id}_{X \wedge Y}$ at the top of the left-hand side. Its image in the bottom left corner is $t_{X,Y}$, and the image of $t_{X,Y}$ under $h_*$ is equal to the image of $h$ under $t^*_{X,Y}$. By definition of $h'$, the image of $\mathsf{id}_{X \wedge Y}$ in the top-right corner is $h$. So the right-hand column sends $h$ to $t^*_{X,Y}(h)$. That is, the composite vertical equivalence is homotopic to $t^*_{X,Y}$. □

2.8. **Smash products, truncation, and suspension.** The goal of this section is to prove a result about the interaction of smash products and truncation, and a result about the interaction of smash products and suspension. Both results make use of the symmetry of the smash product, so we begin with that.

**Definition 2.39.** Given pointed types $X$ and $Y$, there is a pointed map $\tau : X \wedge Y \to_\bullet Y \wedge X$ defined by induction on the smash product in the following way:

- $\tau(\mathsf{sm}(x, y)) :\equiv \mathsf{sm}(y, x)$;
- $\tau(\mathsf{auxl}) :\equiv \mathsf{auxr}$;
- $\tau(\mathsf{auxr}) :\equiv \mathsf{auxl}$;
- $\mathsf{ap}_\tau(\mathsf{gluel}\ y) := \mathsf{gluer}\ y$.
- $\mathsf{ap}_\tau(\mathsf{gluer}\ x) := \mathsf{gluel}\ x$.

It is pointed by $\mathsf{refl}_{\mathsf{sm}(y_0, x_0)}$.

**Lemma 2.40.** *For pointed types $X$ and $Y$, the composite $\tau \circ \tau : X \wedge Y \to_\bullet X \wedge Y$ is pointed homotopic to the identity. In particular, the map $\tau$ is an equivalence.*

*Proof.* We first show that for every $z : X \wedge Y$, $\tau(\tau(z)) = z$. We prove this using the induction principle for smash products. For the three point-constructors, this holds definitionally. The two 1-dimensional constructors are similar, so we only consider the first one. We must show that for each $y : Y$,

$$\mathsf{transport}^{z \mapsto \tau(\tau(z))=z}(\mathsf{gluel}\ y, \mathsf{refl}_{\mathsf{sm}(x_0,y)}) = \mathsf{refl}_{\mathsf{auxl}}.$$

By a calculation similar to those in [Uni13, Section 2.11], the left-hand-side is equal to

$$\mathsf{ap}_\tau(\mathsf{ap}_\tau(\mathsf{gluel}\ y))^{-1} \cdot \mathsf{refl}_{\mathsf{sm}(x_0,y)} \cdot \mathsf{gluel}\ y.$$

By the definition of $\tau$ in Definition 2.39, this is equal to

$$(\mathsf{gluel}\ y)^{-1} \cdot \mathsf{refl}_{\mathsf{sm}(x_0,y)} \cdot \mathsf{gluel}\ y,$$

which is equal to $\mathsf{refl}_{\mathsf{auxl}}$, as required.

We must also show that this homotopy is pointed. Up to definitional equality, this amounts to showing that $\mathsf{refl}_{\mathsf{sm}(x_0,y_0)} = \mathsf{refl}_{\mathsf{sm}(x_0,y_0)}$, which is true by reflexivity. □

Next we show that the map $\tau$ is natural.

**Lemma 2.41.** *Given pointed maps $f : X \to_\bullet X'$ and $g : Y \to_\bullet Y'$ between pointed types, the following square of pointed maps commutes:*

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{f \wedge g} & X' \wedge Y' \\ \tau \downarrow & & \downarrow \tau \\ Y \wedge X & \xrightarrow{g \wedge f} & Y' \wedge X'. \end{array}$$

*Proof.* By path induction we can reduce to the case that $f(x_0) \equiv x_0'$ and $g(y_0) \equiv y_0'$. Next we use the induction principle for $X \wedge Y$. The square commutes definitionally on the three point constructors of $X \wedge Y$, but requires some straightforward path algebra in the remaining two cases. Since the proof has been formalized, we omit the details. □

**Lemma 2.42.** *Let $m \geq -1$, let $n \geq 0$, let $Y$ be a pointed type, and let $X$ be a pointed, $(n-1)$-connected type. Then the map $\big\| |-|_m \wedge \mathsf{id}_X \big\|_{n+m} : \|Y \wedge X\|_{n+m} \to \big\| \|Y\|_m \wedge X \big\|_{n+m}$ is an equivalence.*

*Proof.* Since the map in the statement is pointed, it is enough to show that for every pointed, $(n+m)$-truncated type $T$, precomposition with $|-|_m \wedge \mathsf{id}_Y$ induces an equivalence

$$(\|Y\|_m \wedge X \to_\bullet T) \longrightarrow (Y \wedge X \to_\bullet T).$$

By the naturality in the first variable of the adjunction from Lemma 2.31, it is enough to show that precomposition with $|-|_m$ induces an equivalence

$$(\|Y\|_m \to_\bullet X \to_\bullet T) \longrightarrow (Y \to_\bullet X \to_\bullet T),$$

and this follows from the fact that the type $X \to_\bullet T$ is $m$-truncated (Lemma 2.27). □

**Corollary 2.43.** *Let $m \geq -1$, let $n \geq 0$, let $X$ be a pointed, $(n-1)$-connected type, and let $Y$ be a pointed type. Then the map $\big\| \mathsf{id}_X \wedge |-|_m \big\|_{n+m} : \|X \wedge Y\|_{n+m} \to \big\| X \wedge \|Y\|_m \big\|_{n+m}$ is an equivalence.*

*Proof.* The square

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\mathsf{id}_X \wedge |-|_m} & X \wedge \|Y\|_m \\ \tau \downarrow & & \downarrow \tau \\ Y \wedge X & \xrightarrow{|-|_m \wedge \mathsf{id}_X} & \|Y\|_m \wedge X. \end{array}$$

commutes by Lemma 2.41. The vertical maps are equivalences by Lemma 2.40. By Lemma 2.42, the bottom map is an equivalence after $(n+m)$-truncation, so the top map must also be an equivalence after truncating. □

We conclude this section with a result letting us commute suspension and smash products.

**Lemma 2.44.** *Given pointed types $X$ and $Y$, there is a pointed equivalence*

$$c_\Sigma : \Sigma(X \wedge Y) \simeq_\bullet X \wedge \Sigma Y,$$

*natural in both $X$ and $Y$.*

*Proof.* By Definition 2.39 and Lemmas 2.40 and 2.41, it is enough construct a natural equivalence $\Sigma(X \wedge Y) \simeq_\bullet \Sigma X \wedge Y$. In order to do this, it suffices to show that, for every pointed type $Z$, there is an equivalence $(\Sigma(X \wedge Y) \to_\bullet Z) \simeq (\Sigma X \wedge Y \to_\bullet Z)$ natural in $X$, $Y$, and $Z$, by the Yoneda Lemma (Proposition 2.11). Given a pointed type $Z$, we define the equivalence as the following composite of natural equivalences:

$$\begin{aligned} (\Sigma(X \wedge Y) \to_\bullet Z) &\simeq (X \wedge Y \to_\bullet \Omega Z) \\ &\simeq (X \to_\bullet Y \to_\bullet \Omega Z) \\ &\simeq (X \to_\bullet \Omega(Y \to_\bullet Z)) \\ &\simeq (\Sigma X \to_\bullet Y \to_\bullet Z) \\ &\simeq (\Sigma X \wedge Y \to_\bullet Z). \end{aligned}$$

The first and fourth equivalences follow from the adjunction between suspension and loops (Lemma 2.16). The second and fifth equivalences use Lemma 2.31. The third equivalence follows from Lemma 2.24. This concludes the proof. □

This result was formalized in the Spectral repository [Doo18], but the proof of naturality is not complete.

## 3. Homology and the Hurewicz theorem

In this section, we begin by defining homology and proving the Hurewicz theorem. Then we define the Hurewicz homomorphism and prove that it is unique up to sign. We conclude by giving some applications about the interaction between homology, connectedness, and truncation.

3.1. **Prespectra and homology.** In this section, we introduce prespectra as a tool for defining the homology groups of a type.

**Definition 3.1.** A **prespectrum** $(Y, s)$ is a family of pointed types $Y : \mathbb{N} \to \mathcal{U}_\bullet$ and a family of pointed **structure maps** $s : \prod_{(n:\mathbb{N})} Y_n \to_\bullet \Omega Y_{n+1}$. When the structure maps of $Y$ are clear from the context, we will denote the prespectrum simply by $Y$.

**Definition 3.2.** A **map of prespectra** $f : (T, s) \to (T', s')$ consists of a family of pointed maps $f : \prod_{(n:\mathbb{N})} Y_n \to_\bullet Y'_n$, and a family of pointed homotopies $\prod_{(n:\mathbb{N})} \Omega s'_n \circ f_n \sim_\bullet \Omega f_{n+1} \circ s_n$.

Note that a prespectrum can be equivalently defined by giving a family of pointed types $Y : \mathbb{N} \to \mathcal{U}_\bullet$ and a family of pointed maps $\Sigma Y_n \to_\bullet Y_{n+1}$. This is the way that we will specify prespectra.

*Example* 3.3. Eilenberg–Mac Lane spaces are defined in homotopy type theory in [LF14]. Given an abelian group $A$, the **Eilenberg–Mac Lane prespectrum** $HA$ of type $A$ is given by the family $\lambda n.K(A, n)$ of pointed types, where we let $K(A, 0) :\equiv A$, pointed at 0. For $n \geq 1$, the structure map is

$$|-|_{n+1} : \Sigma K(A, n) \longrightarrow \left\|\Sigma K(A, n)\right\|_{n+1} \equiv K(A, n+1).$$

When $n \equiv 0$, we define $\Sigma K(A, 0) \to K(A, 1)$ by induction on suspension, by mapping the north and south poles of $\Sigma K(A, 0)$ to the base point of $K(A, 1)$, and $\mathsf{merid}(a)$ to the loop of $K(A, 1)$ represented by $a$.

**Definition 3.4.** Given a pointed type $X$ and a prespectrum $(Y, s)$, we form a prespectrum $X \wedge Y$, called the **smash product** of $X$ and $Y$, as follows. The type family is given by $(X \wedge Y)_n \equiv X \wedge Y_n$. The structure maps are given by the following composite:

$$\Sigma(X \wedge Y_n) \xrightarrow{c_\Sigma}_\bullet X \wedge \Sigma Y_n \xrightarrow{\mathsf{id}_X \wedge \overline{s_n}}_\bullet X \wedge Y_{n+1}, \tag{3.1}$$

where $\overline{s_n} : \Sigma Y_n \to_\bullet Y_{n+1}$ corresponds to $s_n : Y_n \to_\bullet \Omega Y_{n+1}$.

Note that, by the naturality of Lemma 2.44 and the functoriality of the smash product on pointed types, the smash product of a pointed type and a prespectrum is functorial.

**Definition 3.5.** The type of **sequential diagrams of groups** is the type

$$\mathsf{Grp}^{\mathbb{N}} :\equiv \sum_{A:\mathbb{N}\to\mathsf{Grp}} \prod_{n:\mathbb{N}} A_n \to_{\mathsf{Grp}} A_{n+1}.$$

Analogously, we define the type of **sequential diagrams of abelian groups**, which we denote by $\mathsf{Ab}^{\mathbb{N}}$.

The most important example in this paper is given by sequential diagrams of groups that come from prespectra.

*Example* 3.6. Let $(Y, s)$ be a prespectrum and let $n, k : \mathbb{N}$. The map $s_k : Y_k \to_\bullet \Omega Y_{k+1}$ induces a morphism $\pi_n(s_k) : \pi_n(Y_k) \to_{\mathsf{Grp}} \pi_n(\Omega Y_{k+1}) \simeq \pi_{n+1}(Y_{k+1})$. Iterating this process, we get a sequential diagram of groups $\lambda i.\pi_{n+i}(Y_{k+i}) : \mathbb{N} \to \mathsf{Grp}$. We denote this diagram by $\mathcal{S}^n_k(Y)$. This construction is natural in $Y$.

Note that, if $n \geq 2$, the diagram $\mathcal{S}^n_k(Y)$ is a sequential diagram of abelian groups.

**Definition 3.7.** Let $(A, \varphi) : \mathcal{U}^{\mathbb{N}}$ be a sequential diagram of types, We define the **sequential colimit** of $(A, \varphi)$, denoted by $\mathsf{colim}\, A : \mathcal{U}$, as the higher inductive type generated by the constructors $\iota : \prod_{n:\mathbb{N}} A_n \to \mathsf{colim}\, A$ and $\mathsf{glue} : \prod_{n:\mathbb{N}} \prod_{a:A_n} \iota_n(a) = \iota_{n+1}(\varphi_n(a))$.

**Lemma 3.8.** *Let $(A, \varphi) : \mathsf{Ab}^{\mathbb{N}}$ be a sequential diagram of abelian groups. Then, the sequential colimit $\mathsf{colim}\, A$ of the underlying sets is a set, and it has a canonical abelian group structure such that all of the induced maps $i_n : A_n \to \mathsf{colim}\, A$ are homomorphisms. Moreover, the abelian group $\mathsf{colim}\, A$ has the universal property of the colimit in the category of abelian groups.*

*Proof.* The main difficulty is to show that $\mathsf{colim}\, A$ is 0-truncated. For this, we use [DRS20, Corollary 7.7 (1)], which says that a sequential colimit of $n$-truncated types is $n$-truncated.

To show that $\mathsf{colim}\, A$ has an abelian group structure we start by using induction to define the operation $+$ on $\mathsf{colim}\, A$. In the case of point constructors, we define $\iota_l(a) + \iota_n(b) :\equiv \iota_m(\varphi_l^m(a) + \varphi_n^m(b))$, where $m \equiv \max(l, n)$ and $\varphi_l^m : A_l \to A_m$ and $\varphi_n^m : A_n \to A_m$ are defined by iterating $\varphi$. The case of a path constructor $\mathsf{glue}$ and a point constructor is straightforward, and the case of two path constructors is immediate, since $\mathsf{colim}\, A$ is a set. The fact that, with these operation, $\mathsf{colim}\, A$ is an abelian group is clear.

The map $\iota_n : A_n \to \mathsf{colim}\, A$ is a group morphism for every $n$ by construction, and the fact that $\mathsf{colim}\, A$ satisfies the universal property of the colimit follows from the induction principle of $\mathsf{colim}\, A$. □

**Definition 3.9.** Let $Y$ be a prespectrum, let $n : \mathbb{Z}$, and let $j \equiv \max(0, 2 - n)$. We define the ***n*-th stable homotopy group** of $Y$ as

$$\pi_n^s(Y) :\equiv \mathsf{colim}\, \mathcal{S}_j^{n+j}(Y).$$

Note that the stable homotopy groups of a prespectrum are defined for any integer $n$, and not only for natural numbers. Moreover, by construction, the sequential diagram in the definition of $\pi_n^s(Y)$ is a sequential diagram of abelian groups, so stable homotopy groups are always abelian. As an aside, one can show that any $\mathcal{S}_j^{n+j}(Y)$ with $j \geq \max(0, 2 - n)$ will have an isomorphic colimit. Finally, since the construction $\mathcal{S}_k^n(Y)$ is functorial in $Y$, stable homotopy groups are functorial in the prespectrum.

**Definition 3.10.** We define the $n$-th **reduced homology** of $X$ with coefficients in $Y$ as

$$\tilde{H}_n(X; Y) :\equiv \pi_n^s(X \wedge Y).$$

We define the $n$-th **(ordinary) reduced homology** of $X$ with coefficients in an abelian group $A$ by

$$\tilde{H}_n(X; A) :\equiv \tilde{H}_n(X; HA).$$

Notice that these types carry an abelian group structure, given by the group structure of stable homotopy groups (Definition 3.9).

3.2. **The Hurewicz theorem.** In this section, we prove our main result, Theorem H. To do so, we first show that when $X$ is sufficiently connected, we can compute $\tilde{H}_n(X; A)$ without taking a colimit.

**Lemma 3.11.** *Let $n \geq 1$, let $A : \mathsf{Ab}$, and let $X$ be a pointed, $(n-1)$-connected type. Then the natural homomorphism $\pi_{n+1}(X \wedge K(A, 1)) \to \tilde{H}_n(X; A)$ is an equivalence.*

*Proof.* Recall that $\tilde{H}_n(X; A) \equiv \pi_n^s(X \wedge HA) \equiv \mathsf{colim}\, \mathcal{S}_j^{n+j}(X \wedge HA)$, for $j = \max(0, 2-n)$. Since $n \geq 1$, we must consider two cases, $n = 1$ and $n \geq 2$. When $n = 1$, we have $j = 1$, and the sequential diagram that defines $\tilde{H}_n(X; A)$ starts as follows:

$$\pi_{n+1}(X \wedge K(A, 1)) \longrightarrow \pi_{n+2}(X \wedge K(A, 2)) \longrightarrow \cdots$$

When $n \geq 2$, we have $j = 0$, and the sequential diagram that defines $\tilde{H}_n(X; A)$ starts as follows:

$$\pi_n(X \wedge K(A, 0)) \longrightarrow \pi_{n+1}(X \wedge K(A, 1)) \longrightarrow \pi_{n+2}(X \wedge K(A, 2)) \longrightarrow \cdots$$

It suffices to show that in either case the morphism

$$\pi_{n+i}(X \wedge K(A, i)) \longrightarrow \pi_{n+i+1}(X \wedge K(A, i+1))$$

is an equivalence for $i \geq 1$. To prove this, we use Eq. (3.1) to factor the map as

$$\pi_{n+i}(X \wedge K(A,i)) \longrightarrow \pi_{n+i+1}(X \wedge \Sigma K(A,i)) \longrightarrow \pi_{n+i+1}(X \wedge K(A,i+1)).$$

Now, the first of these two maps is induced by the Freundenthal map $X \wedge K(A,i) \to \Omega\Sigma(X \wedge K(A,i))$ composed with the equivalence $\Sigma(X \wedge K(A,i)) \simeq (X \wedge \Sigma K(A,i))$. Notice that, by Corollary 2.32, $X \wedge K(A,i)$ is $(n+i-1)$-connected. If $i \geq 1$, we have that $n+i \geq 2$, and thus $(n+i-1)+1 \leq 2(n+i-1)$, so the Freudenthal suspension theorem ([Uni13, Theorem 8.6.4]) implies that the map $\pi_{n+i}(X \wedge K(A,i)) \to \pi_{n+i+1}(X \wedge \Sigma K(A,i))$ is an equivalence.

The second map is an equivalence by Corollary 2.43, since by definition $K(A,i+1) \equiv \left\|\Sigma K(A,i)\right\|_{i+1}$. ☐

**Theorem 3.12** (Hurewicz Theorem)**.** *Given an abelian group $A$, a natural number $n \geq 1$, and a pointed, $(n-1)$-connected type $X$, we have an isomorphism $\pi_n(X)^{ab} \otimes A \simeq_{\mathsf{Grp}} \tilde{H}_n(X;A)$, natural in $X$ and $A$.*

By naturality in $X$, we mean naturality with respect to pointed maps between $(n-1)$-connected types.

*Proof.* By Lemma 3.11, it is enough to show that we have a natural isomorphism $\pi_{n+1}(X \wedge K(A,1)) \simeq_{\mathsf{Grp}} \pi_n(X)^{ab} \otimes A$, and this follows directly from Theorem 2.38. ☐

3.3. **The Hurewicz homomorphism.** In this section we give a construction of the Hurewicz homomorphism and prove that it is unique up to sign.

Let $X$ be a pointed type, $A$ an abelian group, and $n \geq 1$. Applying $\tilde{H}_n(-;A)$ to the $(n-1)$-connected cover map $X\langle n-1\rangle \to_\bullet X$ we obtain a morphism $\tilde{H}_n(X\langle n-1\rangle;A) \to_{\mathsf{Grp}} \tilde{H}_n(X;A)$, natural in $X$ and $A$. By Theorem 3.12, there is a natural isomorphism

$$\pi_n(X\langle n-1\rangle)^{ab} \otimes A \simeq_{\mathsf{Grp}} \tilde{H}_n(X\langle n-1\rangle;A).$$

Since $\pi_n(X\langle n-1\rangle) \to_{\mathsf{Grp}} \pi_n(X)$ is also a natural isomorphism, we can compose with the abelianization of its inverse to obtain a morphism $\pi_n(X)^{ab} \otimes A \to_{\mathsf{Grp}} \tilde{H}_n(X;A)$.

**Definition 3.13.** For every $X : \mathcal{U}_\bullet$, $A : \mathsf{Ab}$, and $n \geq 1$, the morphism $h_n : \pi_n(X)^{ab} \otimes A \to \tilde{H}_n(X;A)$ described above is the ***n*th Hurewicz homomorphism**.

By construction, when $X$ is $(n-1)$-connected, $h_n$ is an isomorphism.

**Definition 3.14.** Let $n \geq 1$. A morphism of ***n*-Hurewicz type** is given by a group homomorphism $\pi_n(X)^{ab} \otimes A \to_{\mathsf{Grp}} \tilde{H}_n(X;A)$ for each $X : \mathcal{U}_\bullet$ and $A : \mathsf{Ab}$, that is natural in both $A$ and $X$, and that is an isomorphism when $X \equiv S^n$ and $A \equiv \mathbb{Z}$. Here we are regarding $\pi_n(-)^{ab} \otimes -$ and $\tilde{H}_n(-;-)$ as 1-coherent functors $\mathsf{U}_\bullet \times \mathsf{Ab} \to \mathsf{Ab}$.

*Example* 3.15. For any $n \geq 1$, the $n$th Hurewicz homomorphism (Definition 3.13) is a morphism of $n$-Hurewicz type.

**Theorem 3.16.** *Let $n : \mathbb{N}$ and let $F$ and $G$ be morphisms of $n$-Hurewicz type. Then either $F(X,A) = G(X,A)$ or $F(X,A) = -G(X,A)$ for every pointed type $X$ and abelian group $A$. The choice of sign is independent of $X$ and $A$.*

*Proof.* The morphisms $F(S^n,\mathbb{Z})$ and $G(S^n,\mathbb{Z})$ give us two isomorphisms between $\pi_n(S^n) \otimes \mathbb{Z}$ and $\tilde{H}_n(S^n;\mathbb{Z})$. We now show that there are exactly two possible isomorphisms between $\pi_n(S^n) \otimes \mathbb{Z}$ and $\tilde{H}_n(S^n;\mathbb{Z})$, and that these differ by a sign. On the one hand, by [LS13] (see also [Uni13, Section 8.1]), we know $\pi_n(S^n) \simeq \mathbb{Z}$. On the other hand, we have $\mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z}$ (Example 2.36). So it is enough to show that there are exactly two isomorphisms between $\mathbb{Z}$ and $\mathbb{Z}$, and that they differ by a sign. This is straightforward, using the fact that if two integers $n$ and $m$ satisfy $n \times m = 1$, then $n = m = 1$ or $n = m = -1$, which follows from the fact that $\mathbb{Z}$ has decidable equality.

There are then two cases, $F(S^n, \mathbb{Z}) = G(S^n, \mathbb{Z})$ and $F(S^n, \mathbb{Z}) = -G(S^n, \mathbb{Z})$. We consider only the first case, the second one being analogous. We thus assume that $F(S^n, \mathbb{Z}) = G(S^n, \mathbb{Z})$ and we want to show that for every pointed type $X$ and every abelian group $A$ we have $F(X, A) = G(X, A)$.

By Lemma 2.37, it is enough to check that $F(X, A) = G(X, A)$ when evaluated on elementary tensors. Since the abelianization map is surjective and we are proving a proposition, it is enough to check this on elementary tensors $(\eta\alpha) \otimes \beta$ for $\alpha : \pi_n(X)$ and $\beta : A$. Since we are proving a mere proposition, we can assume that we have a pointed map $\overline{\alpha} : S^n \to_\bullet X$ representing $\alpha$. Define $\overline{\beta} : \mathbb{Z} \to A$ by sending 1 to $\beta$. Consider the following diagram, which commutes by the naturality assumption:

$$\begin{array}{ccc}
\pi_n(S^n)^{ab} \otimes \mathbb{Z} & \xrightarrow{F(S^n, \mathbb{Z})} & \tilde{H}_n(S^n; \mathbb{Z}) \\
\Big\downarrow {\scriptstyle \pi_n(\overline{\alpha})^{ab} \otimes \overline{\beta}} & & \Big\downarrow {\scriptstyle \tilde{H}_n(\overline{\alpha}, \overline{\beta})} \\
\pi_n(X)^{ab} \otimes A & \xrightarrow{F(X, A)} & \tilde{H}_n(X; A).
\end{array}$$

The commutativity of the diagram implies that

$$F(X, A)((\eta\alpha) \otimes \beta) = \tilde{H}_n(\overline{\alpha}, \overline{\beta})(F(S^n, \mathbb{Z})(\theta \otimes 1)),$$

where $\theta : \pi_n(S^n)$ is represented by the identity map $S^n \to_\bullet S^n$. Similarly, we get that $G(X, A)((\eta\alpha) \otimes \beta) = \tilde{H}_n(\overline{\alpha}, \overline{\beta})(G(S^n, \mathbb{Z})(\theta \otimes 1))$, and since $F(S^n, \mathbb{Z}) = G(S^n, \mathbb{Z})$, we conclude that $F(X, A)(\alpha \otimes \beta) = G(X, A)(\alpha \otimes \beta)$. □

3.4. **Applications.** In this section, we give some consequences of the main results in the paper. We start with two immediate applications of the Hurewicz theorem.

**Proposition 3.17.** *Let $n \geq 1$, let $X$ be a pointed, $n$-connected type, and let $A : \mathsf{Ab}$. Then $\tilde{H}_i(X; A) = 0$ for all $i \leq n$. Conversely, if $X$ is a pointed, connected type with abelian fundamental group such that $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \leq n$, then $X$ is $n$-connected.* □

**Proposition 3.18.** *Let $n \geq 1$ and let $A, B : \mathsf{Ab}$. Then $\tilde{H}_n(K(A, n); B) \simeq A \otimes B$, and, in particular, $\tilde{H}_n(K(A, n); \mathbb{Z}) \simeq A$.* □

The following result says that truncation does not affect low-dimensional homology.

**Proposition 3.19.** *Let $X$ be a pointed type and let $m \geq n$ be natural numbers. For every abelian group $A : \mathsf{Ab}$, the truncation map $X \to \lVert X \rVert_m$ induces an isomorphism $\tilde{H}_n(X; A) \xrightarrow{\sim} \tilde{H}_n\left(\lVert X \rVert_m; A\right)$.*

*Proof.* The objects in the sequential diagram that defines $\tilde{H}_n(X; A)$ have the form $\pi_{n+i}(X \wedge K(A, i))$ for $i \geq \min(0, 2 - n)$, and the morphism $\tilde{H}_n(X; A) \to \tilde{H}_n(\lVert X \rVert_m; A)$ is induced by levelwise morphisms $\pi_{n+i}(X \wedge K(A, i)) \to \pi_{n+i}(\lVert X \rVert_m \wedge K(A, i))$ given by the functoriality of $\pi_{n+i}$ and the smash product. We will show that these levelwise morphisms are isomorphisms, which implies that the induced map is an isomorphism.

Consider the commutative square

$$\begin{array}{ccc} \pi_{n+i}(X \wedge K(A,i)) & \longrightarrow & \pi_{n+i}\left(\left\|X\right\|_m \wedge K(A,i)\right) \\ \downarrow & & \downarrow \\ \pi_{n+i}\left(\left\|X \wedge K(A,i)\right\|_{i+m}\right) & \longrightarrow & \pi_{n+i}\left(\left\|\left\|X\right\|_m \wedge K(A,i)\right\|_{i+m}\right), \end{array}$$

given by functoriality of $(i+m)$-truncation and $\pi_{n+i}$. It suffices to show that the bottom map and the vertical maps in the square are isomorphisms. The vertical maps are isomorphisms since $n+i \leq i+m$, and the bottom map is an isomorphism by Lemma 2.42. ☐

We conclude by showing that $\infty$-connected maps induce an isomorphism in all homology groups.

**Corollary 3.20.** *Let $f : X \to_\bullet Y$ be a pointed map between pointed types that induces an isomorphism in $\pi_0$ and an isomorphism in $\pi_n$ for $n \geq 1$ and all choices of basepoint $x_0 : X$. Then $f$ induces an isomorphism in all homology groups for all choices of coefficients.*

*Proof.* Let $A : \mathsf{Ab}$ and let $n \geq 0$. We have a commutative square

$$\begin{array}{ccc} \tilde{H}_n(X;A) & \longrightarrow & \tilde{H}_n(Y;A) \\ \downarrow & & \downarrow \\ \tilde{H}_n\left(\left\|X\right\|_n;A\right) & \longrightarrow & \tilde{H}_n\left(\left\|Y\right\|_n;A\right), \end{array}$$

where the vertical maps are isomorphisms, by Proposition 3.19. The bottom map is induced by $\left\|f\right\|_n : \left\|X\right\|_n \to \left\|Y\right\|_n$, which is an equivalence, by the truncated Whitehead theorem ([Uni13, Theorem 8.8.3]). It follows that the top map is an isomorphism, concluding the proof. ☐

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