No set of spaces detects isomorphisms in the homotopy category

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Introduction

Recall the following fundamental result:

Theorem (J.H.C. Whitehead, 1949). Let $f : X \to Y$ be a pointed map between pointed, connected CW complexes. TFAE:

- f is a pointed homotopy equivalence.
- For $n > 0$, the induced map $[S^n, X]_\bullet \to [S^n, Y]_\bullet$ is a bijection.

The $\bullet$ means that we are considering pointed homotopy classes of pointed maps, so $[S^n, X]_\bullet = \pi_n(X)$.

Question. Does there exist a set $\mathcal{G}$ of spaces such that for any map $f : X \to Y$ between connected CW complexes, TFAE:

- f is a homotopy equivalence.
- For each $S \in \mathcal{G}$, the induced map $[S, X] \to [S, Y]$ is a bijection.

Here we are considering unpointed homotopy classes.

History

Question. Does there exist a set $\mathcal{G}$ of spaces such that for any map $f : X \to Y$ between ~~connected~~ CW complexes, TFAE:

- f is a homotopy equivalence.
- For each $S \in \mathcal{G}$, the induced map $[S, X] \to [S, Y]$ is a bijection.

In 1981, Heller claimed that the answer to the above question is "No." However, his proof relied on an "obvious" fact, which turns out to be false. To explain this, we need some definitions.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{G} \subseteq \mathcal{C}$ be a set of objects.

- The set $\mathcal{G}$ jointly reflects isomorphisms if a morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism whenever $\mathcal{C}(S, f) : \mathcal{C}(S, X) \to \mathcal{C}(S, Y)$ is a bijection for every $S \in \mathcal{C}$.

With this terminology, the question is whether Ho(Top) contains a set which jointly reflects isomorphisms.

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- A weak colimit of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a cocone W through which every cocone Z factors, not necessarily uniquely.

- The set $\mathcal{G}$ is bounded if for each sufficiently large regular cardinal κ, every diagram $D : \kappa \to \mathcal{C}$ has a weak colimit W such that

$$\operatorname*{colim}_{\alpha<\kappa} \mathcal{C}(S, D(\alpha)) \longrightarrow \mathcal{C}(S, W)$$

is a bijection for every $S \in \mathcal{G}$.

$$\begin{tikzcd}
D(0) \arrow[r] & D(1) \arrow[r] & D(2) \arrow[r] & \cdots \arrow[r] & W \\
 & & & & S \arrow[u, "\forall"'] \arrow[ulll, dashed, "\exists\ \text{(almost unique)}"]
\end{tikzcd}$$

History II

Theorem (Heller, 1981). Assume that $\mathcal{C}$ has coproducts and weak pushouts, and contains a **bounded** set $\mathcal{G}$ of objects which **jointly reflects isomorphisms**. Let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. TFAE:

(i) F is representable, i.e., $F(-) \cong \mathcal{C}(-, K)$ for some $K \in \mathcal{C}$.

(ii) F sends coproducts and weak pushouts in $\mathcal{C}$ to products and weak pullbacks in Set.

Example (Heller). There is a functor $\mathrm{Ho}(\mathrm{Top})^{\mathrm{op}} \to \mathrm{Set}$ which satisfies (ii) but is not representable.

Claim (Heller). Every set $\mathcal{G}$ of CW complexes is **bounded** in Ho(Top).

Conclusion (Heller). There does not exist a set $\mathcal{G}$ of objects which **jointly reflects isomorphisms** in Ho(Top).

Our results (C.-Arlin/Carlson)

Theorem 1. The set $\mathcal{G} = \{S^n\}$ of spheres in Ho(Top) is **not bounded**.

This **reopens** the question given at the start of this talk.

Theorem 2. There does not exist a set $\mathcal{G}$ of spaces that **jointly reflects isomorphism** in Ho(Top).

So Heller was right!

In contrast, in the **2-category** of spaces, maps, and homotopy classes of homotopies, the spheres **do** jointly reflect isomorphisms. (Arlin/Raptis)

Methods

Theorem 1 is much harder to prove than Theorem 2. We show:

Theorem 1. For every uncountable regular cardinal κ, there is a diagram $D : \kappa \to \mathrm{Ho}(\mathrm{Top})$ which has no weak colimit W satisfying

$$\operatorname*{colim}_{\alpha<\kappa} \mathcal{C}(S^n, D(\alpha)) \cong \mathcal{C}(S^n, W) \quad \forall n > 0. \tag{1}$$

We first show that it is sufficient to find a counterexample in the homotopy category of groupoids, where maps are functors up to natural isomorphism.

Given κ, the diagram D we consider sends α to the free group on α generators, with morphisms the natural inclusions.

We then use Serre's theory of graphs of groups and Higgins' work on their fundamental groupoids to construct a sufficiently pathological cocone, which we use to show that D has no weak colimit satisfying (1).

Methods II

Theorem 2. There does not exist a set $\mathcal{G}$ of spaces that **jointly reflects isomorphism** in $\mathrm{Ho}(\mathrm{Top})$.

The proof of Theorem 2 requires no special tools. Here's a sketch.

Let $\mathcal{G}$ be a set of spaces and let α be a regular cardinal larger than $|\pi_1(S)|$ for each $S \in \mathcal{G}$.

Let $\Sigma_\alpha^{\mathrm{c}}$ denote the group of bijections $\sigma : \alpha \to \alpha$ which move fewer than α elements.

Define the shift homomorphism $s : \Sigma_\alpha^{\mathrm{c}} \to \Sigma_\alpha^{\mathrm{c}}$ by

$$s(\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1, & \gamma = \gamma' + 1 \\ \gamma, & \gamma \text{ a limit ordinal.} \end{cases}$$

Our example will be $Bs : B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$.

Methods III

We're considering $Bs : B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$ where

$$s(\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1, & \gamma = \gamma' + 1 \\ \gamma, & \gamma \text{ a limit ordinal.} \end{cases}$$

Recall that $\mathrm{Ho}(\mathrm{Top})(BG, BH) \cong \mathrm{Hom}(G, H)/\text{conjugation by } H$. Therefore, Bs is a homotopy equivalence iff s is an isomorphism. But s is not surjective, since $s(\sigma)$ always preserves limit ordinals. Therefore, $Bs : B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$ is not a homotopy equivalence.

Claim: $(Bs)_* : \mathrm{Ho}(\mathrm{Top})(S, B\Sigma_\alpha^{\mathrm{c}}) \to \mathrm{Ho}(\mathrm{Top})(S, B\Sigma_\alpha^{\mathrm{c}})$ is a bijection for every $S \in \mathcal{G}$.

It suffices to prove this for $S = BG$, where G is any group of cardinality less than α. We in fact show:

Claim 2: $(Bs)_* : \mathrm{Ho}(\mathrm{Top})(BG, B\Sigma_\alpha^{\mathrm{c}}) \to \mathrm{Ho}(\mathrm{Top})(BG, B\Sigma_\alpha^{\mathrm{c}})$ is **the identity map** for every G of cardinality less than α.

Methods IV (if time)

Claim 2: $(Bs)_* : \mathrm{Ho}(\mathrm{Top})(BG, B\Sigma_\alpha^{\mathrm{c}}) \to \mathrm{Ho}(\mathrm{Top})(BG, B\Sigma_\alpha^{\mathrm{c}})$ is **the identity map** for every G of cardinality less than α.

Any map $BG \to B\Sigma_\alpha^{\mathrm{c}}$ is induced by a homomorphism $\phi : G \to \Sigma_\alpha^{\mathrm{c}}$.
Let $\beta < \alpha$ be a cardinal such that $\phi(g) \in \Sigma_\beta$ for all $g \in G$.
Then $s \circ \phi$ is conjugate to ϕ via $\tau \in \Sigma_\alpha^{\mathrm{c}}$ defined by

$$\tau(\gamma) = \begin{cases} \gamma - 1, & \gamma < \beta \text{ a successor ordinal} \\ \beta + \gamma, & \gamma < \beta \text{ a limit ordinal} \\ \gamma + 1, & \beta \leq \gamma < \beta + \beta \\ \gamma, & \text{otherwise.} \end{cases}$$

Some straightforward calculations show:

- τ is a bijection.
- τ fixes $\gamma \geq \beta + \beta$, so $\tau \in \Sigma_\alpha^{\mathrm{c}}$.
- $\tau^{-1}\sigma\tau = s(\sigma)$ for $\sigma \in \Sigma_\beta$.

In particular, $s \circ \phi$ is conjugate to ϕ, and Claim 2 follows. $\square$

References

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